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THE GEOMETRY OF THE WEDDLE MANIFOLD W_p.

By ARTHUR B. COBLE and JOSEPHINE H. CHANLER.

Introduction. The Weddle manifold W_p has been defined [1] to be that manifold of p dimensions ($p \geqq 2$) in an odd space S_{2p-1} which is the locus of fixed points of a certain Cremona involution I attached to a symmetric set of $2p+2$ F-points, P_{2p+2}^{2p-1}. The unique rational norm-curve, N^{2p-1}, on P_{2p+2}^{2p-1} serves as a convenient reference curve for points of the space S_{2p-1}. The importance of W_p is due not merely to its intrinsic geometric interest but also to the fact that W_p is birationally related to the generalized Kummer manifold K_p of Klein and Wirtinger which is defined by the theta squares provided the theta functions of genus p are of *hyperelliptic* type. The primary purpose of this memoir is to study the geometry of W_p itself, but the matters chosen for study are sometimes such as are fundamentally related to the mapping of W_p upon K_p.

It has been shown [1] that the coördinates of a point on W_p can be expressed by means of hyperelliptic theta functions, and that then the theta squares determine on W_p the sections of W_p by the members of a certain mapping system Σ of order p with $(p-1)$-fold points at P_{2p+2}^{2p-1}. This system Σ maps W_p upon K_p provided the dimension of Σ is 2^p-1, the dimension of the space of K_p. In § 1 the complete base of Σ is obtained, the dimension 2^p-1 of Σ is verified, and the dimensions of the subsystems of Σ which contain F-spaces of W_p of various kinds are found. When $p>2$ these subsystems yield "singular spaces" of K_p of novel type.

Algebraic parametric representations of the generic point on W_p are given in § 2, and these are extended in §§ 3, 4, . . . to study certain systems of curves on W_p. A sketch of the content appears in ([5]).

1. The mapping system Σ. We recall [cf. [1], § 3] the *finite Cremona group*, $G_{2^{2p+1}}$, attached to the figure P_{2p+2}^{2p-1} of $2p+2$ points in S_{2p-1}, say $p_1, \cdots, p_{2p+2}$. If $p_3, \cdots, p_{2p+2}$ are taken as reference points, p_1, p_2 as points y, z, the equations of the element I_{12} of $G_{2^{2p+1}}$ are $x_i x'_i = y_i z_i$ $(i=1, \cdots, 2p)$. The abelian $G_{2^{2p+1}}$ is generated by elements I_{ij} of this type. We recall also the definition of the F-loci of the elements of this group—in particular the k-th F-locus of the j-th kind, $\pi^{(j)}_{i_1 i_2 \ldots i_{2k+2-j}}$ $(j=1, \cdots, p)$. This is, when $j \leqq k \leqq (2p+j)/2$, the locus of dimension $2p-1-j$ which is described by the ∞^{k-j} S_{2p-k-1}'s on $p_{i_{2k+3-j}}, \cdots, p_{i_{2p+2}}$ and on $k-j$ variable points of

N^{2p-1}, the norm-curve on P^{2p-1}_{2p+2}; and the locus has the order $\binom{k}{j}$, the multiplicity $\binom{k}{j}$ on the $S_{2p-2k+j-1}$ defined by $p_{i_{2k+3-j}}, \cdots, p_{i_{2p+2}}$, and the multiplicity $\binom{k-1}{j}$ along N^{2p-1}. On the other hand, when $(j-2)/2 \gtreqless k < j$, it is the locus of dimension $j-1$ which is described by the ∞^{j-k-1} S_k's on $p_{i_1}, \cdots, p_{i_{2k+2-j}}$ and on $j-k-1$ variable points of N^{2p-1}; and this locus has the order $\binom{2p-k-1}{j-k-1}$, the multiplicity $\binom{2p-k-1}{j-k-1}$ on the S_{2k-j+1} defined by $p_{i_1}, \cdots, p_{i_{2k+2-j}}$, and the multiplicity $\binom{2p-k-2}{j-k-2}$ along N^{2p-1}.

All the F-loci of the j-th kind are conjugate under $G_{2^{2p+1}}$. When $j = p$, they all are of the same dimension $p-1$. When however $j < p$, those for which $k \geqq j$ have the dimension $2p-1-j$, and those for which $k < j$ have the smaller dimension $j-1$. With respect to these F-spaces of smaller dimension we state the theorem:

(1) *The mapping system Σ, the system of spreads of order p with $(p-1)$-fold points at P^{2p-1}_{2p+2}, contains every F-locus for which $k < j < p$ as a basic locus of multiplicity $p-j$. No other points of S_{2p-1} are base points of Σ.*

The first case of this theorem, $k = 0$, $j = 1$ restates the defining property of Σ, i. e., that Σ has $(p-1)$-fold points at P^{2p-1}_{2p+2}. The second case, $k = 0$, $j = 2$, states that Σ contains N^{2p-1} as a basic locus of multiplicity $p-2$. We first prove the theorem for this case. Let $(\alpha x)^p = 0$ be a generic member of Σ. We observe then that

(1.1) *$(\alpha x)^p = 0$ contains N^{2p-1} if $p > 2$.*

For, α cuts N^{2p-1} at P^{2p-1}_{2p+2} in $(2p+2)(p-1) = (2p-1)p + (p-2)$ points, whence α contains N^{2p-1} if $p-2 \geqq 1$.

We now prove the lemma:

(1.2) *If $q_1, \cdots, q_t$ are any t points $(t \gtreqless p-3)$ of P^{2p-1}_{2p+2}, then $\phi_t(x) \equiv (\alpha q_1)(\alpha q_2) \cdots (\alpha q_t)(\alpha x)^{p-t} = 0$ has $(p-t)$-fold points at $q_1, \cdots, q_t$, $(p-t-1)$-fold points at the remaining points of P^{2p-1}_{2p+2}, and contains N^{2p-1}.*

For, the lemma is true for $t = 1$, since $\phi_1(x) = (\alpha q_1)(\alpha x)^{p-1} = 0$ has a $(p-1)$-fold point at q_1 with the same tangent cone as $(\alpha x)^p = 0$ (thus, according to (1.1), containing the tangent to N^{2p-1} at q_1) and has $(p-2)$-fold points at the remaining points of P^{2p-1}_{2p+2}. Thus $\phi_1(x)$ contains N^{2p-1} if $p \geqq 4$ since it meets N^{2p-1} at P^{2p-1}_{2p+2} in $(2p+1)(p-2) + p = (p-1)(2p-1) + (p-3)$ points. Since the lemma is true for $t = 1$, let us assume that it is true for values of t up to $t-1$, i. e., that $\phi_{t-1}(x) \equiv (\alpha q_1) \cdots (\alpha q_{t-1})(\alpha x)^{p-t+1}$

$=0$ has $(p-t+1)$-fold points at $q_1, \cdots, q_{t-1}$, has $(p-t)$-fold points at $q_t, \cdots, q_{2p+2}$, and contains N^{2p-1}. Then $\phi_t(x) = (\alpha q_1) \cdots (\alpha q_t)(\alpha x)^{p-t} = 0$ has $(p-t)$-fold points at $q_1, \cdots, q_t$, $(p-t-1)$-fold points at $q_{t+1}, \cdots, q_{2p+2}$, and has at q_t the same tangent cone as $\phi_{t-1}(x)$. Hence $\phi_t(x)$ touches N^{2p-1} at q_t, and, by virtue of its symmetry, at $q_1, \cdots, q_{t-1}$ also. Thus $\phi_t(x) = 0$ meets N^{2p-1} at P^{2p-1}_{2p+2} in $t(p-t) + t + (2p+2-t)(p-t-1) = (p-t)(2p-1) + (p-t-2)$ points. Hence $\phi_t(x)$ contains N^{2p-1} if $p-t-2 \geqq 1$, or if $t \leqq p-3$.

The proof of (1.2) being thus complete, we observe (for $t = p-3$) that $(\alpha q_1) \cdots (\alpha q_{p-3})(\alpha x)^3 = 0$ contains N^{2p-1}, whence

(1.3) *The third polar of any point on N^{2p-1} as to any member of Σ is apolar to any $p-3$ points of P^{2p-1}_{2p+2}.*

To complete the proof of (1) for the case $k=0$, $j=2$, i. e., that

(1.4) *Every member of Σ contains N^{2p-1} to multiplicity $p-2$ at least,*

we take $2p$ points of the set P^{2p-1}_{2p+2} as reference points. Then $(\alpha x)^p$ has the form $\Sigma a_{i_1 \ldots i_p} x_{i_1} \ldots x_{i_p}$ since the reference points are $(p-1)$-fold. If y is any point on N^{2p-1}, the polar $(\alpha y)^3 (\alpha x)^{p-3}$ has the form $\Sigma b_{i_1 \ldots i_{p-3}} x_{i_1} \ldots x_{i_{p-3}} = 0$. But, according to (1.3), every $b_{i_1 \ldots i_{p-3}}$ is zero; whence $(\alpha y)^3 (\alpha x)^{p-3} \equiv 0$ in x; i. e. y on N^{2p-1} is a $(p-2)$-fold point.

The basic loci of Σ, $\pi^{(j)}_{(k)}$, mentioned in (1) are defined by the inequalities,

$$(1.5) \qquad (j-2)/2 \leqq k < j \leqq p-1.$$

All the F-loci of the j-th kind constitute a conjugate set under $G_{2^{2p+1}}$; and in such a set the basic F-loci are distinguished from the others by the fact that their dimension is $j-1$ rather than $2p-j-1$. It is convenient to represent these loci by the notation,

$$(1.6) \qquad \pi^{(j)}_{(k)} = S_k(q^{2k+2-j} z^{j-k-1}),$$

which indicates a locus of S_k's on some selected set of $2k+2-j$ points q in P^{2p-1}_{2p+2} and on $j-k-1$ variable points z on N^{2p-1}.

We now examine the generic point on $\pi^{(j)}_{(k)}$ in (1.6) which can be represented as

$$z = \lambda_1 q_1 + \cdots + \lambda_{2k+2-j} q_{2k+2-j} + \mu_1 z_1 + \cdots + \mu_{j-k-1} z_{j-k-1}.$$

This is a $(p-j)$-fold point of $(\alpha x)^p = 0$ if $(\alpha z)^{j+1} (\alpha x)^{p-j-1} \equiv 0$. This $(j+1)$-th polar of z has the form,

$$\Sigma c (\alpha q_1)^{r_1} \cdots (\alpha q_{2k+2-j})^{r_{2k+2-j}} (\alpha z_1)^{s_1} \cdots (\alpha z_{j-k-1})^{s_{j-k-1}} (\alpha x)^{p-j-1},$$

where $r_1 + \cdots + r_{2k+2-j} + s_1 + \cdots + s_{j-k-1} = j + 1$. Since $(\alpha q_i)^2 (\alpha x)^{p-2} \equiv 0$ [cf. (1)], and $(\alpha z_i)^3 (\alpha x)^{p-3} \equiv 0$ [cf. (1.4)], a term of this polar vanishes unless in it every $r_i \eqslantless 1$ and every $s_i \eqslantless 2$. For the non-vanishing terms, therefore, $\Sigma r_i + \Sigma s_i \eqslantless (2k + 2 - j) + 2(j - k - 1) = j$. Since the sum must be $j + 1$, there are no non-vanishing terms. Thus the proof of the multiplicity statement in (1) is complete.

In order to prove that Σ has no other basis points, we observe first that

(1.7) *All of the basic F-loci of Σ in* (1) *are contained in the basic F-loci of the kind $j = p - 1$.*

For, if j' ($< j$) and k' satisfy the inequalities (1.5), and if we take $k' = j' - a$ ($a \eqslantgtr 1$), then we can take $k = j - a$. The F-locus $S_{k'}(q^{2k'+2-j'} z^{j'-k'-1})$ is then contained in the F-locus $S_k(q^{2k+2-j} z^{j-k-1})$. Indeed $2k' + 2 - j' = j' - 2a + 2 < 2k + 2 - j = j - 2a + 2$, and $j' - k' - 1 = a - 1 = j - k - 1$. Hence each $S_{k'}$ on the one locus is contained in an S_k on the other locus. Thus the basic F-loci of kind j' are all contained on those of kind $j = p - 1$ of maximum dimension. We have thus only to prove that every basis point of Σ is on a basic F-locus of kind $j = p - 1$.

Included among the F-loci are those of the first kind, $j = 1$. These have a somewhat exceptional position. For $k = 0$, they are basic, being the sets of directions about each of the points of P^{2p-1}_{2p+2}. For larger values of $k \eqslantless p$ they are the P-loci, or *principal manifolds*, of the elements of the Cremona $G_{2^{2p+1}}$. They are paired in such wise that the members of a pair make up one of 2^{2p} degenerate members of the mapping system Σ [cf. [1], § 5, (31a)]. We have listed these pairs in the table (1.8) below. Opposite them are listed the basic F-loci for $j = p - 1$. We prove that Σ has no other basis points by showing that the F-loci listed are the only points common to all the degenerate members of Σ that are listed.

The table is as follows:

$$
\begin{array}{lrll}
 & & \textit{Degenerate members of } \Sigma & \textit{Basic loci } (j = p - 1) \\
 & 1 : & S_{p-1}(q^1 z^{p-1}) & 1 : S_{p-2}(r^{p-1}) \\
 & 2 : & S_p(q^3 z^{p-2}) \cdot S_{2p-2}(q'^{2p-1}) & 2 : S_{p-3}(r^{p-3} z) \\
 & & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 & k : & S_{p-2+k}(q^{2k-1} z^{p-k}) & \\
(1.8) & & \quad \cdot S_{2p-k}(q'^{2p+3-2k} z^{k-2}) & l : S_{p-1-l}(r^{p+1-2l} z^{l-1}) \\
 & & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 & (p+1)/2 : & S_{(3p-3)/2}(q^p z^{(p-1)/2}) & \\
 & & \quad \cdot S_{(3p-1)/2}(q'^{p+2} z^{(p-3)/2}) & (p+1)/2 : S_{(p-3)/2}(z^{(p-1)/2}) \\
 & (p+2)/2 : & S_{(3p-2)/2}(q^{p+1} z^{(p-2)/2}) & \\
 & & \quad \cdot S_{(3p-2)/2}(q'^{p+1} z^{(p-2)/2}) & p/2 : S_{(p-2)/2}(r z^{(p-2)/2}).
\end{array}
$$

Here we use the first or the second of the last two lines according as p is odd or even. In the first column q^i is any set of i points selected from P^{2p-1}_{2p+2} and q'^{2p+2-i} is the complementary set of $2p+2-i$ points. In the second column the points r are also selected from P^{2p-1}_{2p+2}. In both columns the z's are variable points on N^{2p-1}.

We observe first that the set 1 of degenerate members of Σ will have in common only the points of $S_{p-2}(z^{p-1})$. For, a point P in S_{2p-1} on $S_{p-1}(p_1 z^{p-1})$ and $S_{p-1}(p_2 z^{p-1})$ is represented by a binary $(2p-1)$-ic which is expressible in two ways as a sum of p $(2p-1)$-th powers. But, there being no identity connecting $2p$ distinct $(2p-1)$-th powers, the coefficients of p_1 and p_2 in the two expressions must vanish, and the points $z_1, \cdots, z_{p-1}$ in the two expressions, as well as their coefficients, must coincide, i. e. P is a point on a $(p-1)$-secant S_{p-2} of N^{2p-1}. We therefore examine for base points only the multi-secant spaces of N^{2p-1} of the dimensions contained in the second column of (1.8) and prove that they are basic only when the number of variable points z is precisely the number indicated.

Consider the degenerate member k of Σ and the basic locus l. This has been shown to be on all of the members of Σ. Consider however the $S_{p-1-l}(r^{p-2l}z^l)$, which arises from l by changing one fixed r to a variable z, with reference to k. If $k-l$ of the points r are found in q, the remaining $p-k-l$ points r and l points z can be found among the $p-k$ variable points z of the first factor of k and $S_{p-1-l}(r^{p-2l}z^l)$ is contained on this first factor. If however only $k-l-1$ of the points r are in q, the S_{p-1-l} is not contained in the first factor. The remaining $p-k-l+1$ points r are already contained among the points q' of the second factor, but the $k-2$ points z of the second factor can not be so disposed as to include the first $k-l-1$ points r and the l variable points z so that $S_{p-1-l}(r^{p-2l}z^l)$ is not contained on certain k's and is therefore not basic. Hence the basis S_{p-1-l} has at most $l-1$ variable points on N^{2p-1} as in (1.8) and the proof of (1) is complete.

(2) *The dimension of the linear system Σ of spreads of order p with $(p-1)$-fold points at P^{2p-1}_{2p+2} in S_{2p-1} is 2^p-1.*

We take as coördinate system in S_{2p-1} the coefficients of a binary $(2p-1)$-ic, $(\alpha t)^{2p-1} \equiv (\alpha' t)^{2p-1} \equiv \cdots$. Perfect powers such as $(tt_1)^{2p-1}$ then determine the points t_1 on N^{2p-1}; and in particular $t = t_1, \cdots, t_{2p+2}$ determine the points of P^{2p-1}_{2p+2} on N^{2p-1}. We set

$$(2.1) \qquad (\omega t)^{2p+2} = (tt_1)\cdot(tt_2)\cdots(tt_{2p+2}).$$

A spread of order p is determined by a form,

$$(2.2)\qquad f(s_1^{2p-1}s_2^{2p-1}\cdots s_p^{2p-1}) \equiv (\beta_1 s_1)^{2p-1}(\beta_2 s_2)^{2p-1}\cdots(\beta_p s_p)^{2p-1},$$

symmetric in the sets of binary variables $s_1,\cdots,s_p$ and of order $2p-1$ in each set. A point represented by the binary $(2p-1)$-ic above is on this spread if the apolarity condition,

$$(2.3)\qquad f(\alpha^{2p-1}\alpha'^{2p-1}\cdots\alpha^{(p-1)2p-1}) \equiv (\beta_1\alpha)^{2p-1}(\beta_2\alpha')^{2p-1}\cdots(\beta_p\alpha^{(p-1)})^{2p-1}=0,$$

is satisfied. If the binary forms, $(\alpha t)^{2p-1}$, $(\alpha' t)^{2p-1},\cdots$ are regarded as distinct, the condition (2.3) expresses that the corresponding p distinct points are apolar to the p-ic spread; and in particular the vanishing of (2.2) expresses that the points $t=s_1,\cdots,s_p$ on N^{2p-1} are apolar to the p-ic.

We observe first that

(2.4) *A symmetric form, $f(s_1^r s_2^r\cdots s_j^r)$, is uniquely determined by its linear covariant, $f(s^r s^r s_3^r\cdots s_j^r)$, to within a symmetric form, $g(s_1^{r-2j+2}\cdots s_j^{r-2j+2})\cdot\Pi(s_m s_n)^2$ $[m<n=1,\cdots,j]$, where g is a generic symmetric form of the orders indicated.*

For, if F_1, F_2 are two symmetric forms with this linear covariant, then $(F_1-F_2)_{s_1=s_2=s}\equiv 0$. Hence F_1-F_2 contains the factor (s_1s_2). The residual factor must change sign if s_1, s_2 interchange, whence it also contains a factor (s_1s_2). Thus F_1-F_2, being symmetric, contains the symmetric factor, $\Pi(s_m s_n)^2$, and the residual factor is any symmetric factor of the form g.

(2.5) *A generic symmetric form, $f(s_1^r s_2^r\cdots s_j^r)$ contains $\binom{j+r}{r}$ linearly independent coefficients.*

For, it can be interpreted as above as a spread of order j in S_r.

(2.6) *The necessary and sufficient condition that the symmetric form* (2.2) *represent a member of Σ is that*

$$f(s^{2p-1}s^{2p-1}s_3^{2p-1}\cdots s_p^{2p-1}) \equiv (\omega s)^{2p+2}\cdot(ss_3)^2\cdots(ss_p)^2\cdot g(s_3^{2p-3}\cdots s_p^{2p-3}),$$

where g is a symmetric form of the orders indicated.

For, f in (2.2) being symmetric, it represents a spread of order p, and f in (2.6) represents the second polar of s on N^{2p-1} as to this spread. Since the points P^{2p-1}_{2p+2} are $(p-1)$-fold on a member of Σ, this polar must vanish identically in $s_3,\cdots,s_p$ when $s=t_1,\cdots,t_{2p+2}$, and thus the factor $(\omega s)^{2p+2}$ must occur. Conversely,

(a) the symmetry of f in (2.2), and

(b) the occurrence of the factor $(\omega s)^{2p+2}$ in f in (2.6), ensure that f in (2.2) is a member of Σ. Moreover, if in (2.6) we think of $s, s_4, \cdots, s_p$ as given, then f is the equation in variable s_3 of the linear polar of $s, s, s_4, \cdots, s_p$ on N^{2p-1}. Since N^{2p-1} is a $(p-2)$-fold curve on Σ, it is a simple curve on the cubic polar of $s_4, \cdots, s_p$; and the linear polar of s on N^{2p-1} as to this cubic touches N^{2p-1} at s, whence the factor $(ss_3)^2$ occurs.

There still remains the proper determination of g in (2.6) and this determination must arise entirely from (2.6) and the symmetry mentioned in (a) above. However, according to (2.4), this g in (2.6) is independent of a generic term in f in (2.2) of the form

$$(2.7) \qquad g(s_1{}^1 s_2{}^1 \cdots s_p{}^1) \cdot \Pi(s_m s_n)^2$$

with $p+1$ linearly independent terms. From (a) and (2.6) there follows that

$$f(s_1{}^{2p-1} s_2{}^{2p-1} r^{2p-1} r^{2p-1} s_5{}^{2p-1} \cdots s_p{}^{2p-1})$$
$$\equiv (\omega r)^{2p+2} \cdot (rs_1)^2 (rs_2)^2 (rs_5)^2 \cdots (rs_p)^2 \cdot g(s_1{}^{2p-3} s_2{}^{2p-3} s_5{}^{2p-3} \cdots s_p{}^{2p-3}).$$

Setting $s_1 = s_2 = s$ in this, and setting $s_3 = s_4 = r$ in (2.6), and comparing the right members, we find that

$$(2.8) \qquad g(r^{2p-3} r^{2p-3} s_5{}^{2p-3} \cdots s_p{}^{2p-3})$$
$$= (\omega r)^{2p+2} \cdot (rs_5)^2 \cdots (rs_p)^2 \cdot h(s_5{}^{2p-5} \cdots s_p{}^{2p-5}),$$

where h is a symmetric form of the orders indicated. It is to be observed that

$$f(s^{2p-1} s^{2p-1} r^{2p-1} r^{2p-1} s_5{}^{2p-1} \cdots s_p{}^{2p-1})$$
$$= (\omega r)^{2p+2} \cdot (\omega s)^{2p+2} \cdot (rs)^4 (rs_5)^2 \cdots (rs_p)^2 (ss_5)^2 \cdots (ss_p)^2 \cdot h$$

makes complete use of the symmetry of f in $s_1, \cdots, s_4$ so far as coincidences are concerned, since if three of the variables coincide, f vanishes identically, N^{2p-1} being a $(p-2)$-fold curve of the spread f.

Again h in (2.8) is conditioned by (2.6), but, in passing from g to h, there remains according to (2.4) undetermined in g, a generic form,

$$(2.9) \qquad g(s_3{}^3 s_4{}^3 \cdots s_p{}^3) \cdot \Pi(s_m s_n)^2 \qquad [m < n = 3, \cdots, p]$$

which may be taken at random with $\binom{p+1}{3}$ linearly independent terms.

An entirely similar argument applied to h, on setting $s_5 = s_6 = u$, yields

$$(2.10) \qquad h(u^{2p-5} u^{2p-5} s_7{}^{2p-5} \cdots s_p{}^{2p-5})$$
$$= (\omega u)^{2p+2} \cdot (us_7)^2 \cdots (us_p)^2 \cdot i(s_7{}^{2p-7} \cdots s_p{}^{2p-7}),$$

h being determined by this form i to within a form,

$$(2.11) \qquad g(s_5{}^5, \cdots, s_p{}^5) \cdot \Pi(s_m s_n)^2 \qquad [m < n = 5, \cdots, p],$$

with $\binom{p+1}{5}$ linearly independent coefficients. On continuing this process we find [cf. (2.7), (2.9), (2.11)] that the requirements (a), (b) determine f in (2.2) only to within $\binom{p+1}{1} + \binom{p+1}{3} + \binom{p+1}{5} + \cdots = 2^p$ linearly independent arbitrary coefficients which completes the proof of (2).

This determination of the system Σ suggests the following coördinate system for members of Σ. Let the symmetric form (2.2) which represents a member of Σ be denoted by $f_p{}^{(2p-1)}$. Let $\Pi_1{}^p\Omega_{ij}{}^2$ be the operator which, operating on $f_p{}^{(2p-1)}$, produces $g_p{}^{(1)} = (\beta_1 s_1) \cdots (\beta_p s_p) \cdot \Pi_1{}^p(\beta_i\beta_j)^2$. We have then a sequence of symmetric forms f, and a symmetric covariant g of each, namely:

$$(2.12) \quad \begin{array}{ll}
f_p{}^{(2p-1)} & : \Pi_1{}^p\Omega_{ij}{}^2(f_p{}^{(2p-1)}) \equiv g_p{}^{(1)} \\
[f_p{}^{(2p-1)}]_{s_p=s_{p-1}=s} / (\omega s)^{2p+2} \cdot \Pi_1{}^{p-2}(ss_i)^2 \equiv f_{p-2}^{(2p-3)} & : \Pi_1{}^{p-2}\Omega_{ij}{}^2(f_{p-2}^{(2p-3)}) \equiv g_{p-2}^{(3)} \\
[f_{p-2}^{(2p-3)}]_{s_{p-2}=s_{p-3}=s} / (\omega s)^{2p+2} \cdot \Pi_1{}^{p-4}(ss_i)^2 \equiv f_{p-4}^{(2p-5)} & : \Pi_1{}^{p-4}\Omega_{ij}{}^2(f_{p-4}^{(2p-5)}) \equiv g_{p-4}^{(5)} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
[f_3{}^{(p+2)}]_{s_3=s_2=s} / (\omega s)^{2p+2} \cdot (ss_1)^2 \equiv f_1{}^{(p)} & : \qquad f_1{}^{(p)} \equiv g_1{}^{(p)} \\
[f_2{}^{(p+1)}]_{s_2=s_1=s} / (\omega s)^{2p+2} \qquad \equiv f_0{}^{(p-1)} & : \qquad f_0{}^{(p-1)} \equiv g_0{}^{(p+1)}
\end{array}$$

the next to the last, or the last, line being used according as p is odd or even.

Every member of Σ defines uniquely a definite sequence of forms $g_p{}^{(1)}, g_{p-2}^{(3)}, \cdots$, the coefficients of these forms g being 2^p linearly independent combinations of the coefficients of the given member of Σ. Hence

(2.13) *The 2^p arbitrary coefficients of the forms g in* (2.12) *may be taken as the coördinates of a member of the mapping system Σ in* (1).

If the symmetric form $f_{p-2}^{(2p-3)}$ in (2.12) vanishes identically, all the covariants g except $g_p{}^{(1)}$ vanish identically and conversely. But if $[f_p{}^{(2p-1)}]_{s_p=s_{p-1}=s}$ is identically zero for any s, then N^{2p-1} is a $(p-1)$-fold curve, rather than a $(p-2)$-fold curve, on the corresponding member of Σ. If the symmetric form $f_{p-4}^{(2p-5)}$ vanishes identically, all the covariants g except $g_p{}^{(1)}$ and $g_{p-2}^{(3)}$ vanish identically and conversely. Then $[f_p{}^{(2p-1)}]_{s_p=s_{p-1}=s;\, s_{p-2}=s_{p-3}=r}$ vanishes identically for every s and r, or the bisecant locus of N^{2p-1} is a $(p-3)$-fold, rather than a $(p-4)$-fold, locus of the corresponding member of Σ. In general, then

(3) *The necessary and sufficient condition that a member of the mapping*

system Σ *shall belong to the sub-system* $\sigma^{(2i)}$ *of* Σ *which contains the F-locus* $\pi^{(2i)}_{i-1}$ *(the locus of i-secant* S_{i-1}*'s of* $N^{2p-1} = S_{i-1}(z^i)$ [cf. (1.6)]) *to the multiplicity* $p-2i+1$ *rather than* $p-2i$ [cf. (1)] *is* $f^{(2p-1-2i)}_{p-2i} \equiv 0$, *or* $g^{(2i'+1)}_{p-2i'} \equiv 0$ $(p/2 \geqq i' \geqq i \geqq 1)$. *The system* $\sigma^{(2i)}$ *has the dimension* $\binom{p+1}{1} + \binom{p+1}{3} + \cdots + \binom{p+1}{2i-1} - 1$. *The degenerate members of* Σ *listed in the table* (1.8) *which belong to* $\sigma^{(2i)}$ *are the sets* $1, 2, \cdots, i$.

The F-loci of the j-th kind, $j = 2i < p$ are 2^{2p+1} in number, conjugate under $G_{2^{2p+1}}$. However they divide into 2^{2p} pairs, the members of a pair being conjugate under the symmetric element, $I = I_{1,2,\ldots,2p+2}$, in $G_{2^{2p+1}}$. Thus $\pi^{(2i)}$ is paired with $\pi^{(2i)}_{1,2,\ldots,2p+2}$, the locus of $(p-i)$-secant S_{p-i-1}'s of N^{2p-1}, or $S_{p-i-1}(z^{p-i})$. We now prove that if a member of Σ contains $\pi^{(2i)}$ to multiplicity $p-2i+1$ as above, then this member must contain the paired F-locus $\pi^{(2i)}_{1,2,\ldots,2p+2}$ simply, and conversely. For, a point of $\pi^{(2i)}$ is $\lambda_1 r_1 + \cdots + \lambda_i r_i$, and a point of $\pi^{(2i)}_{1,\ldots,2p+2}$ is $\mu_1 s_1 + \cdots + \mu_{p-i} s_{p-i}$, the r's and s's being generic points of N^{2p-1}. Since N^{2p-1} is a $(p-2)$-fold curve on $(\alpha x)^p = 0$, a member of Σ, we have the following identities in x, r, s:

(a) $$(\alpha r_j)^3(\alpha x)^{p-3} \equiv 0, \qquad (\alpha s_k)^3(\alpha x)^{p-3} \equiv 0.$$

If $\pi^{(2i)}$ has multiplicity $p-2i+1$ on $(\alpha x)^p = 0$, we have the identity in x, λ, r:

(b) $$(\alpha, \lambda_1 r_1 + \cdots + \lambda_i r_i)^{2i}(\alpha x)^{p-2i} \equiv 0.$$

Now $\pi^{(2i)}_{1,\ldots,2p+2}$ is contained simply on $(\alpha x)^p = 0$, if

(c) $$(\alpha, \mu_1 s_1 + \cdots + \mu_{p-i} s_{p-i})^p \equiv 0$$

in s and μ. The identity (c) is satisfied in μ if the identity in the $s_1, \cdots, s_{p-i}$,

(d) $$(\alpha s_1)^{k_1}(\alpha s_2)^{k_2} \cdots (\alpha s_{p-i})^{k_{p-i}} \equiv 0 \qquad (k_1 + \cdots + k_{p-i} = p),$$

is satisfied. According to (a) and (b) we need consider only such terms in (d) as have exponents which satisfy

(e) $$k \lesssim 2, \qquad k_1 + k_2 + \cdots + k_i \lesssim 2i - 1$$

for any i of the k's. Suppose that l, m, n of the $p-i$ exponents k have values 0, 1, 2 respectively. Then $l + m + n = p - i$ and $m + 2n = p$ [cf. (d)], whence $n - l = i$. Thus at least i of the k's have the value 2 and (e) cannot be satisfied (the deficiency on the right being one). Therefore (c) is satisfied by virtue of (a) and (b), and $\pi^{(2i)}_{1,\ldots,2p+2}$ is at least a simple locus on

$(\alpha x)^p = 0$. Moreover $\pi^{(2i)}_{1,\ldots,2p+2}$ cannot have a higher multiplicity. Otherwise we should have $l + m + n = p - i$, $m + 2n < p$, $n - l < i$. Thus at most $i - 1$ of the k's must have the value 2 and (e) can be satisfied. Thus certain terms (d) do not necessarily vanish due to (a) and (b).

Conversely, let $(\alpha x)^p = 0$ contain $\pi^{(2i)}_{1,\ldots,2p+2}$ simply. Then (c) and (d) are satisfied and (a) is satisfied as before. We have then to prove that (b) is satisfied, i. e. that

$$\text{(f)} \quad (\alpha r_1)^{l_1} \cdots (\alpha r_i)^{l_i} (\alpha x)^{p-2i} \equiv 0 \qquad (l \gtrless 2;\, l_1 + \cdots + l_i = 2i),$$

or that

$$\text{(g)} \quad (\alpha r_1)^2 \cdots (\alpha r_i)^2 (\alpha x)^{p-2i} \equiv 0.$$

Since $i \gtrless p/2$, we may in (d) let some of the s's coincide, if necessary, to get each r twice and thus would have terms in (d) of the form $(\alpha r_1)^2 \cdots (\alpha r_i)^2 \times (\alpha s_1) \cdots (\alpha s_{p-2i}) = 0$. Since this would be valid for all choices of $s_1, \cdots, s_{p-2i}$ on N^{2p-1}, it would yield (g).

Since all the 2^{2p} pairs of F-spaces of type $\pi^{(2i)}_{1,\ldots,2k}$, $\pi^{(2i)}_{2k+1,\ldots,2p+2}$ are conjugate under $G_{2^{2p+1}}$, we have proved that

(4) *The system* Σ *contains* 2^{2p} *linear sub-systems of type* $\sigma^{(2i)}_{1,\ldots,2k} = \sigma^{(2i)}_{2k+1,\ldots,2p+2} \equiv \sigma^{(2i)}_{(1,\ldots,2k;\,2k+1,\ldots,2p+2)}$, *conjugate under* $G_{2^{2p+1}}$ *and of dimension given in* (3). *The linear system* $\sigma^{(2i)}_{(1,\ldots,2k;\,2k+1,\ldots,2p+2)}$ *is that sub-system of* Σ *which contains the pair of F-loci,*

$$\pi^{(2i)}_{(1,\ldots,2k;\,2k+1,\ldots,2p+2)} \equiv \pi^{(2i)}_{1,\ldots,2k},\ \pi^{(2i)}_{2k+1,\ldots,2p+2},$$

simply, i. e. *to a multiplicity one greater than the normal multiplicity for all members of* Σ.

We have thus far considered only those sub-systems of Σ which contain, to multiplicity one greater than the normal, the F-loci of *even* rank $j = 2i$. These have a greater degree of simplicity due to the fact that for each rank $j = 2i$ there is one system, $\sigma^{(2i)}_{1,2,\ldots,2p+2}$, which is symmetrically related to N^{2p-1}, the corresponding F-loci being loci of multi-secant spaces of N^{2p-1} with no fixed intersections. We now consider the F-loci of odd rank, $F^{(j)}$, $j = 2i - 1$ $(1 \gtrless i \gtrless (p+1)/2)$. As an example of such an $F^{(j)}$ we take $\pi_1{}^{(2i-1)} = S_{i-1}(p_1 z^{i-1})$ which is paired with $\pi^{(2i-1)}_{2,\ldots,2p+2} = S_{p-i}(p_1 z^{p-i})$. The locus $\pi_1{}^{(2i-1)}$ is a basic locus of Σ of multiplicity $p - 2i + 1$ [cf. (1)] except in the end case, (p odd), $i = (p+1)/2$. In this end case the paired loci $\pi_1{}^{(p)}$, $\pi^{(p)}_{2,\ldots,2p+2}$ coincide.

We seek as before the dimension of the linear system $\sigma_1^{(2i-1)}$, contained in Σ, which has $\pi_1^{(2i-1)}$ as a locus of multiplicity $p-2i+2$, one greater than the normal multiplicity of $\pi_1^{(2i-1)}$ on members of Σ. This will be accomplished by discussing the polar system Σ_1 of p_1 with respect to Σ. We give certain preliminary theorems and lemmas which refer to this polarized system Σ_1. A first theorem relating to Σ is

(5) *The linear system Σ of p-ics with $(p-1)$-fold points at P_{2p+2}^{2p-1} has a single member with a p-fold point at p_1.*

The theorem is obvious when $p=2$ and we assume it true for $p=3,\cdots,p-1$. Let M be any member of Σ with a p-fold point at p_1. Carry out on M the involution $I_{2p+1,2p+2}$ of G_2^{2p+1}. Since in general a member of Σ is carried by this I into a member of Σ, this member M with an extra multiplicity at p_1 is transformed into a member M' of Σ which consists of $S_{2p-2}(2,\cdots,2p)$ and a spread $M_{2p-2}[1^{p-1}2^{p-2}\cdots(2p)^{p-2}(2p+1)^{p-1}(2p+2)^{p-1}]^{p-1}$. This M_{2p-2} of order $p-1$ with $(p-1)$-fold points at p_{2p+1}, p_{2p+2}, is a cone with a $(p-1)$-fold line on these two points. It is therefore the dilation from S_{2p-3} of an M_{2p-4} of order $p-1$ with a $(p-1)$-fold point at q_1, and $(p-2)$-fold points at $q_2,\cdots,q_{2p}$. The theorem being true for $p-1$, this M_{2p-4} is unique, whence M_{2p-2}, and M', and therefore M, are unique.

As an immediate consequence of (5) and (2), we have

(5.1) *The dimension of the linear system Σ_1, the polar of p_1 as to Σ, is 2^p-2.*

For, in polarizing, only those members of Σ with a p-fold point at p_1 are lost.

(5.2) *If $(\alpha x)^p=0$ has $(p-2)$-fold points along a norm-curve N^{2p-1}, and has a $(p-1)$-fold point at p_1 on N^{2p-1}, then the polar $(\alpha p_1)(\alpha x)^{p-1}=0$ is a cone of order $p-1$ which contains the tangent to N^{2p-1} at p_1 as a line of $(p-2)$-fold points.*

For, if $(\alpha x)^p=0$ be written as in (2.2) the condition that it have N^{2p-1} as a $(p-2)$-fold curve yields the identity,

$$(\beta_1 s)^{2p-1}(\beta_2 s)^{2p-1}(\beta_3 s)^{2p-1}(\beta_4 s_4)^{2p-1}\cdots(\beta_p s_p)^{2p-1}\equiv 0, \tag{a}$$

in $s, s_4,\cdots,s_p$. The condition that it have a $(p-1)$-fold point at p_1 with parameter $s=t_1$, yields the identity,

$$(\beta t_1)^{2p-1}(\beta_2 t_1)^{2p-1}(\beta_3 s_3)^{2p-1}\cdots(\beta_p s_p)^{2p-1}\equiv 0, \tag{b}$$

in $s_3, \cdots, s_p$. Any point along the tangent to N^{2p-1} at t_1 is given by variable r in the $(2p-1)$-ic, $(t_1 t)^{2p-2} \cdot (rt)$. This point will be a $(p-2)$-fold point of the polar of p_1 if

$$\text{(c)} \quad (\beta_1 t_1)^{2p-1}(\beta_2 t_1)^{2p-2}(\beta_2 r)(\beta_3 t_1)^{2p-2}(\beta_3 r)(\beta_4 s_4)^{2p-1} \cdots (\beta_p s_p)^{2p-1} \equiv 0$$

in $r, s_4, \cdots, s_p$. We wish to prove that (c) is a consequence of (a) and (b). Since all three of these contain $(\beta_4 s_4)^{2p-1} \cdots (\beta_p s_p)^{2p-1}$, we shall omit these factors in the sequel. The identity (a) expresses for arbitrary $s_4, \cdots, s_p$ that the form in s vanishes identically. It vanishes therefore for every k and r in $s = t_1 + kr$. On making this substitution in (a), and taking account of the symmetry in the β's, the coefficient of k^2 yields

$$\begin{aligned} &3\tbinom{2p-1}{2}(\beta_1 t_1)^{2p-1}(\beta_2 t_1)^{2p-1}(\beta_3 t_1)^{2p-3}(\beta_3 r)^2 \\ &+ 3\tbinom{2p-1}{1}^2(\beta_1 t_1)^{2p-1}(\beta_2 t_1)^{2p-2}(\beta_2 r)(\beta_3 t_1)^{2p-2}(\beta_3 r) \equiv 0. \end{aligned}$$

The first term of this vanishes due to (b) on replacing in (b) the arbitrary $s_3{}^{2p-1}$ by $t_1{}^{2p-3}r^2$; the second term is (c).

Another necessary lemma is

(5. 3) *If M is a member of Σ with only a $(p-1)$-fold point at p_1* [cf. (5)], *and if M_1 is the polar of p_1 with respect to M, then any linear S_r on p_1 which is k-fold on M is k-fold on M_1; conversely if an S_{r-1} is k-fold on M_1 and on M, then the $S_r = [S_{r-1}, p_1]$ is k-fold on M.*

It is sufficient to prove this for an $S_1 = yp_1$ on p_1. Let M, M_1 be $(\alpha x)^p = 0$, $(\alpha p_1)(\alpha x)^{p-1} = 0$. Since p_1 is a $(p-1)$-fold point of M, (a) $(\alpha p_1)^2(\alpha x)^{p-2} \equiv 0$. The S_1 is k-fold on M if (b) $(\alpha, y + \lambda p_1)^{p-k+1}(\alpha x)^{k-1} \equiv 0$; or, making use of (a), if (c) $(\alpha y)^{p-k+1}(\alpha x)^{k-1} + (p-k+1)\lambda(\alpha p_1)(\alpha y)^{p-k}(\alpha x)^{k-1} \equiv 0$. This being true for any λ, $(\alpha p_1)(\alpha y)^{p-k}(\alpha x)^{k-1} \equiv 0$, i. e., the polar $(\alpha p_1)(\alpha x)^{p-1}$ has k-fold points at points y on yp_1. Conversely, if y is k-fold on M and M_1, then each term of (c) vanishes, and (b) is satisfied for every point on yp_1.

We now consider the system in Σ with only a $(p-1)$-fold point at p_1. Its dimension is $2^p - 2$ and it contains the basic F-locus $\pi_1{}^{(2i-1)}$ to multiplicity $p - 2i + 1$. The polar system Σ_1 has the order $p-1$ and the following multiplicities: $p-1$ at p_1; $p-2$ at $p_2, \cdots, p_{2p+2}$, along lines $p_1 p_i$ [cf. (5. 3)], and along the tangent to N^{2p-1} at p_1 [cf. (5. 2)]; $p-3$ along N^{2p-1} and on $\pi_1{}^{(3)} = S_1(p_1 z)$; and $p - 2i + 1$ along the basic F-locus $\pi_1{}^{(2i-1)} = S_{i-1}(p_1 z^{i-1})$. Since the system Σ_1 has order $p-1$ and multiplicity $p-1$ at p_1, it is a system of cones defined completely by p_1 and by its section Σ'_1 by an S'_{2p-2} not on p. We examine this system Σ'_1. It has the order $p-1$, the multi-

plicity $p-3$ along the N^{2p-2} which is the projection of N^{2p-1} from p_1 upon S'_{2p-2}, and the multiplicity $p-2$ at the set of points Q^{2p-2}_{2p+2} on N^{2p-2} which is the projection from p_1 of the set P^{2p-1}_{2p+2} on N^{2p-1}. We now prove as for (2.6) that

(5.4) *The necessary and sufficient condition that a symmetric form represent a member of* Σ'_1 *is that*

$$f(s^{2p-2}s^{2p-2}s_3{}^{2p-2}\cdots s^{2p-2}_{p-1}) \equiv (\omega s)^{2p+2}\cdot(ss_3)^2\cdots(ss_{p-1})^2\cdot f_1(s_3{}^{2p-4},\cdots,s^{2p-4}_{p-1})$$

where f_1 *is a symmetric form of the orders indicated.*

This condition utilizes explicitly only that N^{2p-2} is $(p-3)$-fold, and that the points Q^{2p-2}_{2p+2} determined by $(\omega s)^{2p+2}=0$ are $(p-2)$-fold on Σ'_1. The occurrence of the factors $(ss_3)^2,\cdots,(ss_{p-1})^2$ follows as before. In passing from the symmetric form f as in (5.4) to the symmetric form f_1 there is lost, according to (2.4), a symmetric form

(a) $$(s_1s_2)^2\cdots(s_{p-2}s_{p-1})^2\cdot g(s_1{}^2s_2{}^2\cdots s^2{}_{p-1})$$

for which the $\binom{p+1}{2}$ coefficients of g may be taken arbitrarily without affecting the defining properties of the member of Σ'_1 or of f_1. The only conditions on f_1 in (5.4), as in the earlier case (2.8), are those embodied in (5.4), and in the original symmetry, which yield for f_1 the condition,

(b) $$f_1(s^{2p-4}s^{2p-4}s_5{}^{2p-4}\cdots s^{2p-4}_{p-1}) \equiv (\omega s)^{2p+2}\cdot(ss_5)^2\cdots(ss_{p-1})^2\cdot f_2(s_5{}^{2p-6}\cdots s^{2p-6}_{p-1}),$$

and which leave undetermined in f_1 a symmetric form,

(c) $$(s_3s_4)^2\cdots(s_{p-2}s_{p-1})^2\cdot g_1(s_3{}^4,\cdots,s^4{}_{p-1})$$

for which the $\binom{p+1}{4}$ coefficients of g_1 may be taken arbitrarily.

Continuing in this fashion we find that

(5.5) *The dimension of the system* Σ'_1 *in* S'_{2p-2} *of order* $p-1$ *with* $(p-2)$*-fold points at* Q^{2p-2}_{2p+2} *and* $(p-3)$*-fold curve* N^{2p-2} *is* $\binom{p+1}{2}+\binom{p+1}{4}+\cdots-1$ $=2^p-2$.

On comparing this with (5.1) we see that

(5.6) *The polar system* Σ_1 *of* p_1 *as to* Σ *is the conical dilation into* S_{2p-1} *with vertex* p_1 *in* S_{2p-1} *of the system* Σ'_1 *in* S'_{2p-2} *described in* (5.5).

The method of derivation of the dimension of Σ'_1 in (5.5) yields a di-

vision of Σ into sub-systems. First there is the unique member of Σ [cf. (5)] which is lost in the polar system Σ_1 and therefore in the section Σ'_1. This is the sub-system of Σ of dimension $\binom{p+1}{0} - 1$ which contains the F-locus $\pi_1^{(1)} = S_0(p_1)$ to multiplicity p rather than $p-1$. This member is defined by $g \equiv 0$, $g_1 \equiv 0, \cdots$. In passing from any sub-system of Σ'_1 to a sub-system of Σ, this member must be added. Consider next the sub-system of Σ'_1 defined by the identical vanishing of f_1 in (5.4), or by the identical vanishing of the sequence of forms, $g_1 \equiv 0$, $g_2 \equiv 0, \cdots$. If f_1 in (5.4) vanishes identically, the curve N^{2p-2} is $(p-2)$-fold, rather than $(p-3)$-fold, on members of Σ'_1. On applying (5.3), we find that this sub-system of Σ'_1 yields that sub-system of Σ which contains $\pi_1^{(3)} = S_1(p_1 z)$ to multiplicity $p-2$ rather than $p-3$. Its dimension in S'_{2p-2} is $\binom{p+1}{2} - 1$ and thus we find $\binom{p+1}{0} + \binom{p+1}{2} - 1$ as the dimension of the sub-system of Σ which contains $\pi_1^{(3)}$ to multiplicity $(p-3)+1$.

Continuing in this fashion we have the analog of (3) namely:

(6) *The necessary and sufficient condition that a member of* Σ *belong to the sub-system,* $\sigma_1^{(2i-1)}$, *of* Σ *which contains the basic F-locus* $\pi_1^{(2i-1)} = S_{i-1}(p_1 z^{i-1})$ *to the multiplicity* $p-2i+2$ *rather than* $p-2i+1$ *is the identical vanishing of the form* f_{i-1} [cf. (5.4) (b)], *or of the sequence of forms,* $g_{i-1}, g_i, g_{i+1}, \cdots$ [cf. (5.4) (a), (c)], *these forms being determined by the system* Σ'_1 *in* S'_{2p-2}. *The dimension of* $\sigma_1^{(2i-1)}$ *is* $\binom{p+1}{0} + \binom{p+1}{2} + \binom{p+1}{4} + \cdots + \binom{p+1}{2i-2} - 1$.

The F-loci for odd j are also paired into 2^{2p} pairs of type

$$(6.1) \qquad \pi^{(2i-1)}_{(1,\ldots,2k+1;\,2k+2,\ldots,2p+2)} = \pi^{(2i-1)}_{1,\ldots,2k+1},\ \pi^{(2i-1)}_{2k+2,\ldots,2p+2},$$

the members of a pair being conjugate under $I = I_{1,\ldots,2p+2}$, and the 2^{2p} pairs being conjugate under $G_{2^{2p+1}}$. It may be proved by the method preceding (4) that the linear sub-system $\sigma_1^{(2i-1)}$ of Σ which contains the basic locus $\pi_1^{(2i-1)}$ to multiplicity $p-2i+2$ rather than to the normal multiplicity $p-2i+1$ for Σ also contains *simply* the paired F-locus, $\pi^{(2i-1)}_{2,3,\ldots,2p+2}$, which is not basic for Σ, and conversely. We have then the analog of (4), namely:

(7) *The system* Σ *contains* 2^{2p} *linear sub-systems of type,*

$$\sigma^{(2i-1)}_{(1,\ldots,2k+1;\,2k+2,\ldots,2p+2)} = \sigma^{(2i-1)}_{1,\ldots,2k+1} = \sigma^{(2i-1)}_{2k+2,\ldots,2p+2},$$

conjugate under $G_{2^{2p+1}}$, *and of dimension given in* (6). *The linear system* $\sigma^{(2i-1)}_{(1,\ldots,2k+1;\,2k+2,\ldots,2p+2)}$ *is that sub-system of* Σ *which contains the pair of F-loci given in* (6.1) SIMPLY, i. e., *to a multiplicity one greater than the normal multiplicity of the locus for all members of* Σ.

When $j = p$ it appears from the definitions of the F-loci given at the outset that the paired F-loci as defined in (4) and (6.1) coincide. Also these F-loci are not basic for Σ. When $2i = p$ in (3), and $2i - 1 = p$ in (6), the dimensions given both become $2^p - 2$. Thus

(8) *It is a single condition on the members of Σ to contain one of the 2^{2p} F-loci of the p-th kind simply.*

We find in (7) an instance of increased simplicity of statement when the F-loci are brought in paired as in (3) and (6). Another instance is embodied in the theorem:

(9) *A pair of F-loci $\pi^{(j)}$ and a pair of F-loci $\pi^{(j+1)}$ are* INCIDENT *if the division of indices which determines the one can be converted into the division of indices which determines the other by shifting an index from one of the two sets into the other set. Thus a pair $\pi^{(j)}$ contains $2p + 2$ pairs π^{j+1} $(j < p)$, and is contained in $2p + 2$ pairs $\pi^{(j-1)}$ $(j > 1)$.*

Because of the conjugacy of the pairs under $G_{2^{2p+1}}$ it will be sufficient to prove this for one pair for given j and because of the symmetry of the F-loci in the pair it will be sufficient to examine one index in either set. For each value of j' there are linear F-loci, and we take such a typical case, namely:

(a) $\pi^{(j)}_{(1,\ldots,j+2;\, j+3,\ldots,2p+2)} = S_{2p-j-1}(p_{j+3}\cdots p_{2p+2}),\ S_p(p_1\cdots p_{j+2}z^{p-j-1}).$

We compare this with

(b) $\pi^{(j+1)}_{(1,\ldots,j+3;\, j+4,\ldots,2p+2)} = S_{2p-j-2}(p_{j+4}\cdots p_{2p+2}),\ S_p(p_1\cdots p_{j+3}z^{p-j-2}),$

and with

(c) $\pi^{(j-1)}_{(1,\ldots,j+1;\, j+2,\ldots,2p+2)} = S_{2p-j}(p_{j+2}\cdots p_{2p+2}),\ S_p(p_1\cdots p_{j+1}z^{p-j}).$

The first member of (b) is incident with the first member of (a); the second member of (b) is incident with the second member of (a), one z in (a) being fixed at p_{j+3}. The first member of (c) is incident with the first member of (a); the second member of (c) is incident with the second member of (a). Since the shifting of an index from one set to the other can be done in $2p + 2$ ways, the incidences of the theorem are established.

The results obtained in this section lead to certain conclusions with respect to the *hyperelliptic* Kummer manifold K_p in S_{2^p-1}. Since Σ contains members which represent on W_p the theta squares which define K_p, and since the dimension of Σ is $2^p - 1$, then Σ maps W_p upon K_p. In this mapping the pairs of F-loci of the first kind contribute members of the mapping system

which pass into the 2^{2p} *singular spaces of* K_p *of dimension* $2^p - 2$. On the other hand the 2^{2p} F-loci of the p-th kind map into the 2^{2p} *singular points of* K_p [cf. (8) and [1], (66)]. These are the only singular spaces arising from the classic theory. There remain the 2^{2p} pairs of F-loci of kind j $(1 < j < p)$ of W_p. We find in § 3 that these pairs of F-loci meet W_p in manifolds of dimension $p - j$, a property which carries over to K_p. The sub-systems of Σ on a pair of F-loci of kind j yield systems of linear spaces in S_{2^p-1} which have for base a *singular space of* K_p *of the j-th kind.* Thus a translation of the results obtained above is the following:

(10) *The hyperelliptic* K_p *in* S_{2^p-1} *has* p *systems of singular linear spaces* $\Sigma^{(j)}$ $(j = 1, \cdots, p)$, *each system having* 2^{2p} *members. Each space* $\Sigma^{(j)}$ *has in common with* K_p *a manifold of dimension* $p - j$. *The dimension of a linear space* $\Sigma^{(j)}$ *is* $2^p - 2$ *less the dimension of the system* $\sigma^{(j)}$ *as given in* (3) *and* (6). *Each space* $\Sigma^{(j)}$ *contains* $2p + 2$ *spaces* $\Sigma^{(j+1)}$, *and is contained in* $2p + 2$ *spaces* $\Sigma^{(j-1)}$. *The spaces* $\Sigma^{(j)}$ *and spaces* $\Sigma^{(p-j)}$ *are conjugate under the correlation* $G_{2.2^{2p}}$ *of* K_p.

Thus K_3 in S_7 has 4^3 singular S_6's, 4^3 singular S_3's, and 4^3 singular S_0's; K_4 in S_{15} has 4^4 singular S_{14}'s, 4^4 singular S_{10}'s, 4^4 singular S_4's, and 4^4 singular S_0's; etc. These intermediate singular spaces arise from degenerations of loci on the generic K_p. For example, two singular S_6's of K_3 in S_7 meet K_3^{24} in an elliptic E_6 in their common S_5 [cf. [3], p. 188 (3)]. When K_3 is hyperelliptic, this (for proper choice of the two singular S_6's) breaks up into two N^3's with two common points. The two S_3's containing these N^3's are singular S_3's. The two common points of the two N^3's arise from the extra zero of the two thetas which define the two S_6's, these extra zeros being characteristic of the hyperelliptic case.

2. Parametric forms of the hyperelliptic Weddle p-way in S_{2p-1}. The hyperelliptic Weddle p-way in S_{2p-1} has been defined [cf. [1], (34)] as the locus of fixed points of the involution $I = I_{1,\ldots,2p+2}$ in the $G_{2^{2p+1}}$ determined by the set of points P^{2p-1}_{2p+2} in S_{2p-1}. As x varies on W_p, the set of points, P^{2p-1}_{2p+3}, consisting of P^{2p-1}_{2p+2} and x, has been shown to be "associated" to the set of points R^2_{2p+3} which consists of the $2p + 2$ branch points and the multiple point O of a planar hyperelliptic curve H_p of order $p + 2$ with p-fold point at O, and with $2p + 2$ branch lines on O whose parameters are projective to the parameters of P^{2p-1}_{2p+2} on their norm-curve N^{2p-1}. We examine this association.

Let the hyperelliptic curve H_p have the equation,

(1) $$H_p = y_0^2 f_p(y_1, y_2) + 2y_0 f_{p+1}(y_1, y_2) + f_{p+2}(y_1, y_2) = 0,$$

where f_p, f_{p+1}, f_{p+2} are binary forms in y_1, y_2 of orders indicated by the subscripts. We set

$$(2)\qquad (\omega t)^{2p+2} \equiv f^2_{p+1}(t_1 t_2) - f_p(t_1 t_2)\cdot f_{p+2}(t_1 t_2) \equiv (tt_1)(tt_2)\cdots(tt_{2p+2}),$$
$$z_t \equiv \sqrt{(\omega t)^{2p+2}}.$$

In terms of this irrationality, z_t, a parametric equation of H_p is

$$(3)\qquad y_0 : y_1 : y_2 = -f_{p+1}(t_1, t_2) + z_t : t_1 f_p(t_1, t_2) : t_2 f_p(t_1, t_2),$$

the parameter being $t_1 : t_2 = t$. When $(tt_i) = 0$, we have a branch point of the $g_1{}^2$ on H_p with coördinates

$$(4)\qquad y_0 : y_1 : y_2 = -f_{p+1}(t_{i1}, t_{i2}) : t_{i1} f_p(t_{i1}, t_{i2}) : t_{i2} f_p(t_{i1}, t_{i2})$$
$$[i = 1, \cdots, 2p+2].$$

The p-fold point O of H_p has coördinates

$$(5)\qquad y_0 : y_1 : y_2 = 1 : 0 : 0.$$

Thus the $2p+2$ branch points and O, the set $R^2{}_{2p+3}$ in S_2, have as matrix of coördinates (written vertically and with non-homogeneous parameter t) the following:

$$(6)\qquad \begin{array}{cccccc} -f_{p+1}(t_1) & -f_{p+1}(t_2) & -f_{p+1}(t_3) & : & -f_{p+1}(t_{2p+2}) & 1 \\ t_1 f_p(t_1) & t_2 f_p(t_2) & t_3 f_p(t_3) & : & t_{2p+2} f_p(t_{2p+2}) & 0 \\ f_p(t_1) & f_p(t_2) & f_p(t_3) & : & f_p(t_{2p+2}) & 0. \end{array}$$

Using as a coördinate system in S_{2p-1} the coefficients of a $(2p-1)$-ic referred to the N^{2p-1} on P^{2p-1}_{2p+2}, then the $2p+3$ points consisting of P^{2p-1}_{2p+2} and x on W_p have for coördinates:

$$(7)\qquad (tt_1)^{2p-1},\ (tt_2)^{2p-1},\ (tt_3)^{2p-1},\ \cdots,\ (tt_{2p+2})^{2p-1},\ (\alpha t)^{2p-1}.$$

Here the $(2p-1)$-ic, $(\alpha t)^{2p-1}$ is to be determined in such wise that the row product of the row (7), each term with appropriate constant factor, with each of the three rows in (6) is to vanish identically in t, these being the conditions that the two sets of $2p+3$ points be associated [cf. [3], § 13].

We use the notation,

$$(8)\qquad g'_m(s_k) = (s_k s_1)(s_k s_2)\cdots(s_k s_{k-1})(s_k s_{k+1})\cdots(s_k s_m)\quad [k = 1, \cdots, m],$$

in connection with a form $g_m(t) = (ts_1)\cdots(ts_m)$. We also express $f_p(t)$ in factored form as follows:

$$(9)\qquad f_p(t) \equiv (tr_1)(tr_2)\cdots(tr_p).$$

The $(2p+2)$ $2p$-th powers of $(tt_1), \cdots, (tt_{2p+2})$ are related by the identity:

$$\sum_{i=1}^{2p+2} (t_i t)^{2p}/\omega'(t_i) \equiv 0. \tag{10}$$

The $(3p+2)$ $3p$-th powers of $(tt_1), \cdots, (tt_{2p+2})$, $(tr_1), \cdots, (tr_p)$ are related by the identity:

$$\sum_{i=1}^{2p+2} (tt_i)^{3p}/f_p(t_i) \cdot \omega'(t_i) + \sum_{h=1}^{p} (tr_h)^{3p}/\omega(r_h) \cdot f'_p(r_h) \equiv 0. \tag{11}$$

We observe that, due to the identity (10), the row product of (7) and each of the last two rows in (6) is identically zero in t provided that the first $2p+2$ powers in (7) are affected by the following factors respectively:

$$1/f_p(t_1) \cdot \omega'(t_1), \cdots, 1/f_p(t_{2p+2}) \cdot \omega'(t_{2p+2}).$$

With these factors we take the row product of (7) and the first row of (6), and find that

$$(\alpha t)^{2p-1} = \sum_{i=1}^{2p+2} f_{p+1}(t_i) \cdot (tt_i)^{2p-1}/f_p(t_i) \cdot \omega'(t_i). \tag{12}$$

If we polarize the identity (11) with respect to $f_{p+1}(t)$, the first sum yields $(\alpha t)^{2p-1}$ in (12). For the second sum we observe that, in the case of the roots r_h of f_p [cf. (2)],

$$z_{r_h} = f_{p+1}(r_h), \qquad \omega(r_h) = f^2_{p+1}(r_h). \tag{13}$$

Hence this second sum yields

$$-(\alpha t)^{2p-1} = \sum_{h=1}^{p} (tr_h)^{2p-1}/z_{r_h} \cdot f'_p(r_h). \tag{14}$$

This formula shows that the coördinates x on W_p are proportional to abelian functions of $u_1, \cdots, u_p$ on H_p determined by the p-ad of points on H_p:

$$r_1, z_{r_1};\ r_2, z_{r_2};\ \cdots;\ r_p, z_{r_p}; \tag{15}$$

or by its "superposed" p-ad in which the z's all change sign. For, the coefficients α in (14) are symmetric in the p pairs of values r_i, z_{r_i}.

Each value of the parameter t determines a pair of points $t, \pm z_t$ [cf. (3)] on H_p. Thus p values of t, say $r_1, \cdots, r_p$, determine p pairs of points on H_p which can be arranged into 2^p p-ads on H_p with one point of a p-ad from each pair. These 2^p p-ads divide into 2^{p-1} pairs of superposed p-ads and determine 2^{p-1} points x on W_p as in (14). The 2^{p-1} points x are obtained in (14) by taking the changes of sign of $z_{r_1}, \cdots, z_{r_p}$. Hence

(16) *If x is a point of W_p, the p-secant space S_{p-1} of N^{2p-1} on x meets W_p again in $2^{p-1}-1$ remaining points which with x form a conjugate set of 2^{p-1} points under the group of order 2^{p-1} in S_{p-1} which consists of the identity and the harmonic perspectivities determined by opposite spaces of the p-edron in S_{p-1} and on N^{2p-1}.*

As particular cases for $p=2,3,4$ we may mention:

(17) (a) *The bisecant of N^3 on a point $x^{(1)}$ of W_2 in S_3 meets W_2 again in a point $x^{(2)}$ such that $x^{(1)}, x^{(2)}$ are harmonic with the crossings of the bisecant.*

(b) *The trisecant plane of N^5 on a point $x^{(1)}$ of W_3 in S_5 meets W_3 in four points $x^{(1)}, \cdots, x^{(4)}$ whose diagonal triangle is the triad of crossings of the trisecant plane.*

(c) *The quadri-secant S_3 of N^7 on a point $x^{(1)}$ of W_4 in S_7 meets W_4 in eight points $x^{(1)}, \cdots, x^{(8)}$ which make up with the four crossings of S_3 a set of desmic tetrahedra in the S_3.*

We observe also that

(18) *The $(2p-1)$-ic in (14) which represents with respect to N^{2p-1} the point x on W_p defined by H_p in (1) is the $(2p-1)$-ic which is apolar to f_p and f_{p+1}.*

For, the form of the $(2p-1)$-ic in (14) indicates its apolarity with $f_p=(tr_1)\cdots(tr_p)$ [cf. (9)]. If also we operate on the $(2p-1)$-ic with f_{p+1}, and make use of (13), the result vanishes identically by virtue of the linear relation among the $(p-2)$-th powers of $(tr_1), \cdots, (tr_p)$.

In general this $(2p-1)$-ic is not also apolar to f_{p+2}. It will be, however, if

(19) $$g_0 f_{p+2} - 2g_1 f_{p+1} + g_2 f_p \equiv 0,$$

where g_0, g_1, g_2 are binary forms in $t_1:t_2$ of the orders indicated. For $p=2$ this identity can be satisfied for any f_4, f_3, f_2. For higher values of p it can be satisfied only if the branch points of H_p are on the conic, H_0:

(20) $$H_0 = g_0 y_0^2 + 2g_1 y_0 + g_2 = 0 \quad [\text{cf. }^3, \S\, 38].$$

The two branch points of H_0 are then on H_p. The line joining these two branch points is $g_0 y_0 + g_1 = 0$. This line cuts H_p^{p+2} in p further points. Eliminating y_0 and using (19), the parameters t of the $p+2$ intersections are given by $g_0(g_1^2 - g_0 g_2) f_p = 0$. The p further points are therefore the

further intersections with H_p of the tangents at the p-fold point. Conversely if the p further intersections of these tangents are on a line and we take this line to be $y_0 = 0$, then $f_{p+2} - (at)^2 \cdot f_p \equiv 0$, and the identity (19) is satisfied for the conic $y_0^2 - (at)^2$. The fundamental $(2p+2)$-ic of branch lines of H_p, and the fundamental quadratic of branch lines of the conic H_0, are now

$$(21) \qquad (\omega t)^{2p+2} = f^2_{p+1} - (at)^2 \cdot f_p^2 = 0, \qquad (at)^2 = 0.$$

In particular for a root t_i of $(\omega t)^{2p+2}$,

$$(22) \qquad f_{p+1}(t_i)/f_p(t_i) = \sqrt{(at_i)^2}.$$

We have then, on making use of (12), the theorem:

(23) *If the $2p+2$ branch points of H_p are on a conic, i. e., if the p further intersections of tangents to H_p at the p-fold point are on a line, the point x on W_p determined by H_p is represented by the $(2p-1)$-ic (with reference to N^{2p-1}),*

$$(\alpha t)^{2p-1} = \sum_{i=1}^{2p+2} (tt_i)^{2p-1} \cdot \sqrt{(at_i)^2}/\omega'(t_i).$$

For variable quadratic, $(at)^2$, this point x runs over a manifold $V_2^{(2p-1)}$ on W_p.

The question naturally arises as to whether the signs of the radicals in (23) may be taken at random if the point x is to remain on $V_2^{(2p-1)}$ on W_p. The following lemma shows that the answer is affirmative:

(24) *Given O and a conic H_0. Choose any $2p+2$ points $s_1, \cdots, s_{2p+2}$ on H_0, no one the contact of a tangent from O. Let the line pencil from O to s_i have parameters t_i, and let the line t_i cut H_0 in $s_i(+) = s_i$ and $s_i(-)$. Then there exists an H_p with fundamental $(2p+2)$-ic, t_i, and $2p+2$ branch points s'_i, s'_i being either $s_i(+)$ or $s_i(-)$.*

For, if H_0 is $y_0^2 - y_1y_2 = 0$, or $y_0 : y_1 : y_2 = s : s^2 : 1$, and if the branch points of H_p are on this conic, then $f_{p+2}(y_1, y_2) = y_1y_2f_p(y_1, y_2)$. The H_p can then be written as $(\alpha y)^p \cdot y_0^2 + 2(\beta y)^{p+1}y_0 + y_1y_2 \cdot (\alpha y)^p = 0$, with $2p+3$ homogeneous parameters in the coefficients of the forms $(\alpha y)^p$, $(\beta y)^{p+1}$ in the binary variables y_1, y_2. The curve H_p is on the point $s : s^2 : 1$ of H_0 if $(\alpha s^2)^p \cdot s + (\beta s^2)^{p+1} = 0$. The ratios of the coefficients α, β of this equation in s of degree $2p+2$ are uniquely determined by assigning roots $s_1, \cdots, s_{2p+2}$ to it, and for each choice of s_i or $-s_i$ we have a curve H_p. On the other hand the fundamental $(2p+2)$-ic of H_p is $f^2_{p+1}(y_1, y_2) - y_1y_2f_p^2(y_1, y_2)$

$=f^2_{p+1}(s^2, 1) - s^2 f_p{}^2(s^2, 1)$, and it is independent of the choice of s_i or $-s_i$, since $y_1 : y_2 = s^2 : 1$.

If then in (23) the sign of the radical of $(at_i)^2$ be changed, we have a new point of $V_2{}^{(2p-1}$ which lies with the original point on a line through p_i on N^{2p-1}. Hence

(25) *For variation of the signs of the radicals in* (23) *a closed system of* $2 \cdot 2^{2p}$ *points on* $V_2{}^{(2p-1)}$ *is obtained, the system being projected into itself from each of the points of* P^{2p-1}_{2p+2}. *If the signs of the first two radicals are changed, the point thus obtained is the conjugate of the given point under* I_{12}. *Thus the closed system consists of two sets of* 2^{2p} *points conjugate under the Cremona* $G_2{}^{2p+1}$ *of* W_p, *depending on the parity of the number of changes of sign.*

Here only the second statement in (25) requires additional proof. The involution I_{12} on W_p corresponds in the plane to the quadratic transformation $A_{012} \cdot (12)$, i. e., to the perspective transformation with center O and F-points, O and the first two branch points [cf. [3], § 38]. Then H_p goes into H'_p and s on H_0 into s on H'_0. Thus on H'_0 the $2p$ further branch points have parameters $s_3, \cdots, s_{2p+2}$, but the two fixed branch points have parameters $-s_1$, $-s_2$.

The projection of $V_2{}^{(2p-1)}$ from p_1 upon the $V_2{}^{2p-2}$ in S_{2p-2} determined by a set of points P^{2p-2}_{2p+1} upon an N^{2p-2} with parameters $t_2, \cdots, t_{2p+2}$ is obtained by taking the linear polar of t_1 as to $(\alpha t)^{2p-1}$ in (23). The resulting $(2p-2)$-ic with reference to N^{2p-2} determines the projected point in S_{2p-2}. This $(2p-2)$-ic is

$$(26) \qquad \sum_{i=2}^{2p+2} (t_1 t_i) \cdot (t t_i)^{2p-2} \sqrt{(a t_i)^2} / \omega'(t_i).$$

It has, as is evident, properties entirely analogous to $V_2{}^{(2p-1)}$ and is, due to the loss of one radical, a doubly covered projection. We have thus confirmed analytically the properties of the manifold $V_2{}^{(2p-1)}$ on W_p which were obtained in [[1], § 20 and [2], § 6] geometrically, the $V_2{}^{(2p-1)}$ being defined in the first case as the locus of points in S_{2p-1} from which P^{2p-1}_{2p+2} projects into $2p+2$ points in S_{2p-2} on a rational N^{2p-2}, and in the second case as the locus of nodes of degenerate bi-nodal curves in the family of elliptic norm-curves on P^{2p-1}_{2p+2}.

In this case the generalized theorem, due in the case of W_2 to H. F. Baker, applies not to W_p but rather to $V_2{}^{(2p-1)}$ on W_p.

3. Parametric equations of W_p related to the curves cut out on W_p by $(p+1)$-secant S_p's of N^{2p-1}. In the preceding section we have found

that a generic p-secant S_{p-1} of N^{2p-1} cuts W_p in S_{2p-1} in 2^{p-1} points any one of which is generic on W_p, the others forming with this one a symmetric set [cf. **2** (16)]. We now derive certain expressions for the point $(\alpha t)^{2p-1}$ on W_p and also on the section of W_p by a $(p+1)$-secant S_p of N^{2p-1}. As the dimensions p of the intersecting manifolds indicate, this section is a curve rather than a set of points.

With $(\omega t)^{2p+2} = f^2_{p+1} - f_p f_{p+2}$, and in terms of the factorizations,

$$(1) \qquad \begin{aligned} (\omega t)^{2p+2} &= (tt_1)\cdots(tt_d)\cdots(tt_{2p+2}), \\ f_p(t) &= (tr_1)\cdots(tr_e)\cdots(tr_p), \end{aligned}$$

we have already obtained the following expressions for the point $(\alpha t)^{2p-1}$ on W_p [cf. **2** (12), (13), (14)]:

$$(A) \qquad (\alpha t)^{2p-1} = \sum_{d=1}^{d=2p+2} (tt_d)^{2p-1} \cdot f_{p+1}(t_d)/f_p(t_d) \cdot \omega'(t_d);$$

$$(B) \qquad (\alpha t)^{2p-1} = \sum_{d=1}^{d=2p+2} (tt_d)^{2p-1} \cdot f_{p+2}(t_d)/f_{p+1}(t_d) \cdot \omega'(t_d);$$

$$(C) \qquad -(\alpha t)^{2p-1} = \sum_{e=1}^{e=p} (tt_{r_e})^{2p-1}/f_{p+1}(r_e) \cdot f'_p(r_e).$$

The expression (B) is the same as (A), since, for a root t_d of $(\omega t)^{2p+2}$, $f_{p+1}(t_d)/f_p(t_d) = f_{p+2}(t_d)/f_{p+1}(t_d)$.

Suppose now that the line $y_0 = 0$ in the canonical form of $H_p{}^{p+2}$ is on $j+2$ of the branch points of $H_p{}^{p+2}$ $(j = -2, -1, 0, \cdots, p)$. If $j = -2$, $-1, 0$, this imposes no projective condition on $H_p{}^{p+2}$. If however $j = 1, \cdots, p$, this requires that $H_p{}^{p+2}$ be represented on W_p by a point on an F-locus of the j-th kind. Since $y_0 = 0$ cuts $H_p{}^{p+2}$ in points whose parameters t are given by $f_{p+2}(t) = 0$, the parameters t of these $j+2$ branch points will satisfy both $f_{p+2}(t) = 0$ and $(\omega t)^{2p+2} = 0$, and therefore $f_{p+1}(t) = 0$ also. Let these $j+2$ branch points, say the first $j+2$, be given by $\lambda_{j+2}(t) = 0$. Then we have

$$(2) \qquad \begin{gathered} (\omega t)^{2p+2} = \lambda_{j+2}(t) \cdot \mu_{2p-j}(t), \quad f_{p+2}(t) = \lambda_{j+2}(t) \cdot g_{p-j}(t), \\ f_{p+1}(t) = \lambda_{j+2}(t) \cdot g_{p-j-1}(t). \end{gathered}$$

In addition to the factorizations (1) we introduce also the following:

$$(3) \qquad \begin{aligned} \lambda_{j+2}(t) &= (tt_1)\cdots(tt_a)\cdots(tt_{j+2}), \\ \mu_{2p-j}(t) &= (tt_{j+3})\cdots(tt_b)\cdots(tt_{2p+2}), \\ g_{p-j-1}(t) &= (ts_1)\cdots(ts_c)\cdots(ts_{p-j-1}). \end{aligned}$$

We remove the factor $\lambda_{j+2}(t)$ from the relation, $(\omega t)^{2p+2} = f^2_{p+1} - f_p f_{p+2}$, and obtain

$$(4) \qquad \pi \equiv \mu_{2p-j} - g^2_{p-j-1} \cdot \lambda_{j+2} \equiv - f_p \cdot g_{p-j}.$$

If only the roots $s_1, \cdots, s_{p-j-1}$ of $g_{p-j-1} = 0$ are given, there still remains an undetermined constant factor in g_{p-j-1}, and thus $\pi = 0$ represents a *pencil* of $(2p - j)$-ics, and f_p is a p-ad in some member of the pencil. The given $p - j - 1$ roots s_c of g_{p-j-1}, and the known $j + 2$ roots t_a determine a $(p+1)$-secant S_p of N^{2p-1} to which we may regard the above pencil π as attached.

With f_p, f_{p+1}, f_{p+2} related as above we seek new expressions for $(\alpha t)^{2p-1}$ on W_p. According to (2), for every root t_a of λ_{j+2}, $f_{p+1}(t_a)/f_p(t_a) = 0$; and for every root t_b of μ_{2p-j}, $f_{p+1}(t_b)/f_p(t_b) = f_{p+2}(t_b)/f_{p+1}(t_b) = g_{p-j}(t_b)/g_{p-j-1}(t_b)$. Hence the expressions (A), (B) reduce to

$$\text{(D)} \quad (\alpha t)^{2p-1} = \sum_{b=j+3}^{b=2p+2} (tt_b)^{2p-1} \cdot g_{p-j}(t_b)/g_{p-j-1}(t_b) \cdot \lambda_{j+2}(t_b) \cdot \mu'_{2p-j}(t_b).$$

There is a linear identity connecting the $(3p - 1 - j)$-th powers of the $(3p + 1 - j)$ linear factors, $2p - j$ of which are factors (tt_b) of μ_{2p-j}; $j + 2$, factors (tt_a) of λ_{j+2}; and $p - j - 1$, factors (ts_c) of g_{p-j-1}. This identity, polarized as to g_{p-j}, yields for the powers of (tt_b) the right member of (D). The remaining powers in the identity then yield the following alternative form of $(\alpha t)^{2p-1}$:

$$\text{(E)} \quad -(\alpha t)^{2p-1} = \sum_{a=1}^{j+2} (tt_a)^{2p-1} \cdot g_{p-j}(t_a)/g_{p-j-1}(t_a) \cdot \mu_{2p-j}(t_a) \cdot \lambda'_{j+2}(t_a)$$
$$+ \sum_{c=1}^{p-j-1} (ts_c)^{2p-1} \cdot g_{p-j}(s_c)/\mu_{2p-j}(s_c) \cdot \lambda_{j+2}(s_c) \cdot g'_{p-j-1}(s_c).$$

But, according to (4), for the roots t_a of λ_{j+2}, and the roots s_c of g_{p-j-1}, $g_{p-j}/\mu_{2p-j} = -1/f_p$. Hence

$$\text{(F)} \quad (\alpha t)^{2p-1} = \sum_{a=1}^{j+2} (tt_a)^{2p-1}/f_p(t_a) \cdot g_{p-j-1}(t_a) \cdot \lambda'_{j+2}(t_a)$$
$$+ \sum_{c=1}^{p-j-1} (ts_c)^{2p-1}/f_p(s_c) \cdot \lambda_{j+2}(s_c) \cdot g'_{p-j-1}(s_c).$$

The remainder of this article is devoted to a discussion of these formulae (A), $\cdots$, (F).

When the $(p+1)$-secant S_p of N^{2p-1}, say the $S_p(t_a, s_c)$, is given, the $(\omega t)^{2p+2} = \lambda_{j+2}\mu_{2p-j}$ also being known in advance, the pencil π in (4) is determined, and ∞^1 p-ads f_p of members of the pencil exist, and thus ∞^1 points of W_p are determined. According to (C) such a point is on the

$S_{p-1}(r_e)$, p-secant to N^{2p-1}, and according to (E) it is on the $S_p(t_a, s_c)$. It is thus the unique point common to this S_{p-1} and this S_p.

We have allowed j in (4) to run up to p. The case $j = p$ is quite exceptional, and the case $j = p - 1$ somewhat less so. We examine these two cases in the next section, translating the results obtained to the Kummer K_p. In **5** and **6** we return to the other cases.

4. Sections of W_p by F-loci of the p-th and $(p-1)$-th kind, and singular spaces $\Sigma^{(p)}$ and $\Sigma^{(p-1)}$ of K_p. If we set $j = p$ in the preceding section so that $p + 2$ branch points of H_p^{p+2} are on a line, the pencil π in **3** (4) reduces to the single member, $\mu_p = -f_p$. Thus the p tangents at O are inflexional, and p of the branch points have run up to O. We find in [1, § 3 (16)] that H_p^{p+2} is then represented by any point on the F-locus of the p-th kind, $\pi^{(p)}_{1,2,\ldots,p+2} = \pi^{(p)}_{p+3,\ldots,2p+2}$, which is the S_{p-1} on the last p points of P^{2p-1}_{2p+2}; and that this S_{p-1} is mapped by Σ into one of the *singular points*, $\Sigma^{(p)}_{1,2,\ldots,p+2;\,p+3,\ldots,2p+2}$ of K_p in S_{2^p-1}. To the various points of this S_{p-1} on W_p there correspond on K_p the various directions about the singular point. Thus the 2^{2p} F-loci of the p-th kind of W_p give rise to the 2^{2p} singular points of K_p, $\Sigma_{i_1 i_2 \ldots i_{p+2};\, i_{p+3} \ldots i_{2p+2}}$. The F-loci being conjugate under the Cremona group of W_p, the 2^{2p} singular points of K_p are conjugate under the collineation $g_{2^{2p}}$ of K_p, the map by Σ of the Cremona group.

Again, set $j = p - 1$, so that the $p + 1$ branch points $t_1, \cdots, t_{p+1}$ of H_p^{p+2} are on a line L. The corresponding point of W_p is then on the F-locus, $\pi^{(p-1)}_{1,\ldots,p+1}$, the S_p on $p_{p+2}, \cdots, p_{2p+2}$ of P^{2p-1}_{2p+2}. Under the de Jonquières involution of order $p + 2$ whose locus of fixed points is H_p^{p+2} (which corresponds to $I_{1,2,\ldots,2p+2}$ for which W_p is a locus of fixed points) this line L is transformed into a line M on the $p + 1$ branch points, $t_{p+2}, \cdots, t_{2p+2}$, so that the corresponding point of W_p is also on the paired F-locus, $\pi^{(p-1)}_{p+2,\ldots,2p+2}$, the S_p on $p_1, \cdots, p_{p+1}$. Thus this point must be on the line common to the two S_p's. Conversely, any point on this line is on W_p. For, the pencil **3** (4) is now $\pi = \mu_{p+1} - g_0{}^2 \lambda_{p+1} = -f_p \cdot g_1$. Since g_1 is a variable linear form as g_0 takes all values, and since (D) is linear in the coefficients of g_1, the point $(\alpha t)^{2p-1}$ runs over a line, necessarily the line common to the two S_p's. This line on W_p is mapped by Σ into a rational norm-curve of order p, in the singular space $\Sigma^{(p-1)}_{1,\ldots,p+1;\,p+2,\ldots,2p+2}$ which itself has the dimension p [cf. **1** (4), (7)]. In each of the two S_p's the $p + 1$ points of P^{2p-1}_{2p+2} determine $p + 1$ S_{p-1}'s, F-loci of the p-th kind, each of which meets the line in a point. Such a point maps into a singular point (e. g. $\Sigma^{(p)}_{1,\ldots,p;\,p+1,\ldots,2p+2}$) of K_p on N^p. The $2p + 2$ such singular points on N^p are associated with the linear factors g_1 of

the members of the pencil π for the values $g_0 = 0, \infty$. Thus these factors g_1 are the linear factors of $(\omega t)^{2p+2}$ and the $2p+2$ points on N^p have parameters projective to the roots of $(\omega t)^{2p+2}$. Hence

(1) *A pair of F-loci of W_p of the $(p-1)$-th kind have in common a rational locus which is on W_p itself. These 2^{2p} loci on W_p map into the sections of K_p by its 2^{2p} singular spaces $\Sigma^{(j-1)}$ of dimension p, these sections being rational norm-curves N^p. Each N^p is on $2p+2$ singular points and each singular point is on $2p+2$ N_p's* [cf. **1** (10)]. *On each N^p the parameters of the $2p+2$ singular points are projective to the roots of $(\omega t)^{2p+2} = 0$.*

This is the generalization to K_p of the well-known theorem concerning the incidences of singular points and singular conics of the ordinary Kummer surface K_2 in S_3.

5. Configurations inscribed in the generic curve, the section $[W_p, S_p(t_a, s_c)]$. The section of W_p by the S_p which is $(p+1)$-secant to N^{2p-1} at the $j+2$ points t_a of P^{2p-1}_{2p+2} and at the $p-j-1$ generic points s_c of N^{2p-1} is not usually irreducible. For sufficiently large values of p, the bisecant lines, or the trisecant planes, etc., will be an F-locus of the p-th kind, and therefore will be on W_p. Then the section of W_p by S_p will contain some of these lines, or planes, etc., as the case may be, which are determined by the $p+1$ crossings of N^{2p-1} and S_p. This part of the section will however contain no generic point of W_p. The significant part of the section is the curve attached to the pencil π of **3** (4). For $j = -2, -1, 0$, and fixed t_a, but variable s_c, these curves cover W_p completely. For $j \geqq 1$ they cover completely the section of W_p by an F-locus. We therefore speak of such a curve as the *generic curve of the section* $[W_p, S_p]$.

Reverting to the next to the last paragraph of **3** which states that a point of this curve is cut out on S_p by the S_{p-1} which is p-secant to N^{2p-1} at the p points whose parameters, $f_p = 0$, are a p-ad of a member of the pencil π, we have as an immediate consequence the theorem:

(1) *The pencil π of $(2p-j)$-ics in* **3** (4) *is generic except for the peculiarity that one member contains $p-j-1$ double points. This pencil defines on N^{2p-1} a system of ∞^1 $(2p-j)$-points, each of which determines a* COMPLETE *figure consisting of $\binom{2p-j}{k+1}$ S_k's $(k = 0, 1, \cdots, 2p-j-1)$. The section of these complete figures by $S_p(t_a, s_c)$ yields ∞^1 configurations consisting of $\binom{2p-j}{p+l}$ S_l's $(l = 0, 1, \cdots, p-1$ or $p-j)$. The locus of the ∞^1 sets of $\binom{2p-j}{p}$ points of these configurations is the generic curve of the section*

$[W_p, S_p]$. *The* ∞^1 *configurations of* S_l*'s inscribed in the curve* $[W_p, S_p]$ *are such that an* S_m $(m > l)$ *is on* $\binom{p+m}{m-l}$ S_l*'s, and an* S_l *is on* $\binom{p-j-l}{m-l}$ S_m*'s.*

This is the generalization in the direction both of increasing p and of increasing j of a situation which has been observed by F. Morley and J. R. Conner in the case $p = 2$, $j = -2$. This case has the added interest of indicating that the generic plane section of a Weddle quartic surface in S_3 is not a generic quartic curve.

6. Involution curves $[W_p, S_p]$; sections of W_p by its F-loci. If the members of a pencil of binary n-ics are divided into residual p-ics and $(n-p)$-ics, i.e. if $(\alpha t)^n + \lambda(\beta t)^n = f_p \cdot f_{n-p}$, the p-ics thus obtained constitute an algebraic series (∞^1). Any algebraic curve in one-to-one correspondence with such a system of p-ics will be called an *involution curve,* $I_n^{(p)}$. Since the residual p-ic and $(n-p)$-ic are themselves in corrrespondence an $I_n^{(p)}$ is also an $I_n^{(n-p)}$. The simplest geometric example of $I_n^{(p)}$ is found by plotting binary p-ics in S_p with reference to a norm-curve N^p. Then the coefficients of f_p itself are the coördinates of a point of the space. We shall denote this particular type of involution curve by the symbol $[I_n^{(p)}, N^p]$. The order of this curve is $\binom{n-1}{p-1}$, since an $S_{p-1}(t_1)$ of N^p cuts it in the $\binom{n-1}{p-1}$ points determined by selecting $t_2, \cdots, t_p$ from the n-ic of the pencil which contains t_1.

It is clear from **5** (1) that

(1) *The generic curve of the section,* $[W_p, S_p(t_a, s_c)]$ *is an involution curve,* $I^{(p)}_{2p-j}$.

We wish to examine this involution curve to see in what cases it is the simple type $[I^{(p)}_{2p-j}, N^p]$, and, in other cases, to find its relation to this type. In the formula **3** (E) for a point of this curve we observe that, when the $S_p(t_a, s_c)$ is given, everything in the formula is fixed except first the undetermined constant factor in g_{p-j-1} which runs through the formula and thus may be neglected; and second the coefficients of g_{p-j}, the $(p-j)$-ic factor of a variable $(2p-j)$-ic of the pencil π. Hence this point of the curve $[W_p, S_p]$ varies in S_p only with the variable coefficients of g_{p-j}. In S_p itself the point is expressed linearly in terms of the $p+1$ reference points in S_p, say R_{p+1}, which are respectively:

$$(2) \qquad \begin{aligned} \pi_a &= (tt_a)^{2p-1}/g_{p-j-1}(t_a) \cdot \mu_{2p-j}(t_a) \cdot \lambda'_{(j+2)}(t_a), \\ \pi_c &= (ts_c)^{2p-1}/\mu_{2p-j}(s_c) : \lambda_{j+2}(s_c) \cdot g'_{p-j-1}(s_c). \end{aligned}$$

Thus the parametric equation of the point is

$$(3)\qquad \sum_{a=1}^{j+2} g_{p-j}(t_a)\cdot\pi_a+\sum_{c=1}^{p-j-1} g_{p-j}(s_c)\cdot\pi_c,$$

and the $p+1$ parameters $g_{p-j}(t_a)$, $g_{p-j}(s_c)$ may be taken as the coördinates of the point in S_p referred to R_{p+1}.

If the coefficients of g_{p-j} are taken as point coördinates in S_{p-j} with reference to an underlying N^{p-j} whose points are given by $(tt_1)^{p-j}$, the dual coördinates may also be taken as the coefficients of a $(p-j)$-ic in such wise that the incidence condition is the apolarity condition of two $(p-j)$-ics, the one representing a point and the other an S_{p-j-1}. The hyper-osculating S_{p-j-1}'s of N^{p-j} are then also represented by $(tt_1)^{p-j}$. Thus $g_{p-j}(u_h)$ is the incidence condition of the point g_{p-j} and the S_{p-j-1} of N^{p-j} with parameter u_h. Hence $g_{p-j}(u_1),\cdots,g_{p-j}(u_{p-j+1})$ are the point coördinates of the point g_{p-j} referred to the reference figure R_{p-j+1} formed from the S_{p-j-1}'s of N^{p-j} at $u_1,\cdots,u_{p-j+1}$.

The simplest case is $j=0$. In this case the S_{p-j} of g_{p-j} and N^{p-j} may be identified with the $S_p(t_1t_2s_1\cdots s_{p-1})$ and the point (3) is merely a transform of the point g_{p-j} on an $[I^{(p-j)}_{2p-j}, N^{p-j}]$. The situation is described by the theorem:

(4) *For the case $j=0$, the pencil, $\pi_{2p}=\mu_{2p}-g^2{}_{p-1}\lambda_2=-f_p\cdot g_p$ defines on N^{2p-1} a pencil of $2p$-points, each $2p$-point having $2p$* FACES (i. e., *S_{2p-2}'s on all but one of the $2p$ points). Corresponding to the factorization $\pi_{2p}=f'_1\cdot g'_{2p-1}$ these faces envelop a rational norm-curve K^{2p-1}, a face of K^{2p-1} and the* OPPOSITE *point of the $2p$-point on N^{2p-1} having the same parameter t. The $S_p(t_1t_2s_1\cdots s_{p-1})$ is on the $p-1$ faces of the particular $2p$-point, $g^2{}_{p-1}\lambda_2$, which have parameters $s_1,\cdots,s_{p-1}$. Therefore the faces of K^{2p-1} cut S_p in the S_{p-1}'s of a rational norm-curve N^p in S_p with respect to which g_p determines the point $(\alpha t)^{2p-1}$ in* **3** (E). *The curve $[W_p, S_p(t_1t_2s_1\cdots s_{p-1})]$ is the curve $[I^{(p)}_{2p}, N^p]$ of order $\binom{2p-1}{p-1}$ associated with the g_p's of the above pencil π. On such a section $[W_p, S_p]$ there is an involutorial correspondence set up by the interchange of f_p and g_p.*

The only item in this theorem which requires verification is the identification of N^p as the norm-curve with respect to which g_p is plotted. We examine first the $2p$-ic, μ_{2p}. The face $t_4,\cdots,t_{2p+2}$ with parameter t_3 cuts S_p in an S_{p-1} of N^p with parameter t_3. Thus the faces with parameters $t_3,\cdots,t_{2p+2}$ meet in an $S_{p-1}(t_{p+3},\cdots,t_{2p+2})$ which cuts S_p in a point with parameters $g_p=t_3,\cdots,t_{p+2}$ with reference to N^p. That this is the point on

W_p determined by g_p in **3** (D) is clear, because, if $g_p(t_3), \cdots, g_p(t_{p+2})$ are zero, the point is on the $S_{p-1}(t_{p+3}, \cdots, t_{2p+2})$. We examine also the $2p$-ic, $g^2_{p-1} \cdot \lambda_2 = t_1, t_2, s_1^2, \cdots, s^2_{p-1}$. The faces with parameters $t_2, s_1, \cdots, s_{p-1}$ meet in an $S_{p-1}(t_1 s_1 \cdots s_{p-1})$ which cuts S_p in a point with parameters $g_p = t_2, s_1, \cdots, s_{p-1}$ with reference to N^p. But this is the point t_1 on N^{2p-1} itself. Also in **3** (E) for this g_p all the terms vanish except the term in $(tt_1)^{2p-1}$, and thus the point on W_p coincides with the point determined by g_p with respect to N^p. Thus N^p has in common with the norm-curve attached to g_p at least $3p+1$ S_{p-1}'s, and therefore coincides with it.

The case just discussed separates values $j > 0$ from the two values $j < 0$, *i. e.* $j = -1, -2$. In the latter two cases g_{p-j} is represented on a space of dimension greater than that of S_p; in the former cases on a space of dimension less than that of S_p. Furthermore in these cases $j > 0$ we are dealing only with points of W_p on an F-locus of the j-th kind. These more nearly resemble the case $j = 0$ and we consider them first.

That the faces of the $2p$-points in theorem (4) envelop a rational norm-curve K^{2p-1} is well known. So far as we are aware the corresponding theorem, which applies to the cases $j > 0$ and which is given in (5) is new and we incorporate a proof of it.

(5) *Let there be given in S_n a norm-curve N^n with parameter t and on it ∞^1 r-points defined by the pencil $(\alpha t)^r + k(\beta t)^r = 0$ $[n+1 \geqq r \geqq (n+4)/2]$. The S_{r-1}'s determined by two, and therefore by all of these r-points have a common S_{2r-2-n} $[2r-2-n \geqq 2]$. Each r-point on N^n has r faces, these being S_{r-2}'s on all but one point of the r-point. The r faces of a particular r-point meet this common S_{2r-2-n} in r S_{2r-3-n}'s and the locus of these S_{2r-3-n}'s in S_{2r-2-n} is a rational norm-curve K^{2r-2-n} which is in face-point correspondence with N^n.*

For, it is clear first of all that a particular t_1 determines a particular r-point and that the face of this r-point opposite t_1 is unique. Thus the faces run over a rational locus K. There remains to show that a point in S_{2r-2-n} is on $2r-2-n$ of these faces. If t_1, t belong to the same r-ic of the pencil, they satisfy the symmetric form $(\alpha_1 t_1)^{r-1}(\alpha_2 t)^{r-1} = 0$. For given t_1, this is the $(r-1)$-ic which defines the face t_1, and $(\alpha_1 t_1)^{r-1}(\alpha_2 t)^{r-1} \cdot (tt_1) = 0$ is the r-ic which contains t_1. If $(\gamma t)^n$ represents, with respect to N^n, a point on S_{2r-2-n}, then $(\gamma t)^n$ is apolar to every r-ic of the pencil, i. e.,

$$\text{(a)} \qquad (\alpha_1 t_1)^{r-1}(\alpha_2 \gamma)^{r-1}(\gamma t_1)(\gamma t)^{n-r} \equiv 0 \text{ in } t, t_1.$$

This identity (a) can be replaced by the vanishing of the elementary covariants of (a) whose polars figure in the Clebsch-Gordan development, i. e.

(b) $$(\alpha_2\gamma)^{r-1}(\alpha_1\gamma)^k(\alpha_1 s)^{r-k-1}(\gamma s)^{n-r-k+1} \equiv 0 \qquad (k=0,\cdots,n-r).$$

Similarly the point $(\gamma t)^n$ is on all those faces t_1 for which

(c) $$(\alpha_2\gamma)^{r-1}(\alpha_1 t_1)^{r-1}(\gamma t)^{n-r+1} \equiv 0 \text{ in } t.$$

In this the elementary covariants of the Clebsch-Gordan development are

$$(\alpha_2\gamma)^{r-1}(\alpha_1\gamma)^k(\alpha_1 s)^{r-k-1}(\gamma s)^{n-r-k+1} \qquad (k=0,\cdots,n-r+1).$$

But all of these vanish due to (b) except the last whence (c) has the form

$$k\cdot(\alpha_1 t_1)^{2r-2-n}(\alpha_2\gamma)^{r-1}(\alpha_1\gamma)^{n-r+1}\cdot(t_1 t)^{n-r+1} \equiv 0 \text{ in } t.$$

Thus $(\gamma t)^n$ is on the $2r-2-n$ faces whose parameters t_1 are given by $(\alpha_1 t_1)^{2r-2-n}(\alpha_2\gamma)^{r-1}(\alpha_1\gamma)^{n-r+1}=0$.

We return now to the pencil **3** (4) for values $j=1,\cdots,p-2$. According to **3** (D), (E) the points of W_p determined by the $(p-j)$-ics, g_{p-j}, found in members of the pencil π of $(2p-j)$-ics, lie in the two linear spaces

$$S_{2p-1-j}(t_{j+3},\cdots,t_{2p+2}), \qquad S_p(t_1,\cdots,t_{j+2},s_1,\cdots,s_{p-j-1}),$$

which meet in a space,

(6) $$S_{p-j}.$$

On N^{2p-1} the pencil π defines ∞^1 $(2p-j)$-points to which we apply the lemma (5) by means of the transcription:

$$n=2p-1, \qquad r=2p-j, \qquad S_{2r-2-n}=S_{2p-1-2j}.$$

Thus the pencil π on N^{2p-1} determines an $S_{2p-1-2j}$, and the faces of the $(2p-j)$-points of π cut this $S_{2p-1-2j}$ in the $S_{2p-2-2j}$'s of a rational norm-curve $K^{2p-1-2j}$ in $S_{2p-1-2j}$. The two particular $(2p-j)$-points, defined by μ_{2p-j} and $g^2_{p-j-1}\cdot\lambda_{j+2}$ have parameters $t_{j+3},\cdots,t_{2p+2}$ and $t_1,\cdots,t_{j+2},s_1^2,\cdots,s^2_{p-1-j}$ respectively. Thus the S_{p-j} in (6) is on $S_{2p-1-2j}$. Furthermore, from the particular nature of $g^2_{p-j-1}\cdot\lambda_{j+2}$ in the pencil, this S_{p-j} is on those faces of $K^{2p-1-2j}$ with parameters $s_1,\cdots,s_{p-1-j}$. Thus the faces $K^{2p-1-2j}$ cut S_{p-j} in the S_{p-j-1}'s of a rational norm-curve N^{p-j} in S_{p-j}. Hence

(7) *For the cases $j=1,\cdots,p-2$ the pencil $\pi_{2p-j}=\mu_{2p-j}-g^2_{p-j-1}\cdot\lambda_{j+2}$ $=-f_p\cdot g_{p-j}$ defines on N^{2p-1} a pencil of $(2p-j)$-points whose faces*

cut the S_{p-j} (6) *in the* S_{p-j-1}*'s of a rational norm-curve* N^{p-j} *with respect to which* g^{p-j} *determines the point of* W_p *given by* **3** (E). *The curve* $[W_p, S_p(t_1, \cdots, t_{j+2}, s_1, \cdots, s_{p-j-1})]$ *is the curve* $[I_{2p-j}^{(p-j)}, N^{p-j}]$ *of order* $\binom{2p-j-1}{p-j-1}$ *associated with the* g_{p-j}*'s in* π.

The identification of N^{p-j} with the norm-curve to which g_{p-j} is attached can be carried out as in the case of the theorem (4).

In the F-space of the j-th kind, $S_{2p-1-j}(t_{j+3}, \cdots, t_{2p+2})$ preceding (6), we find ∞^{p-j-1} S_{p-j}'s (6) each containing a curve of the type described in (7) whence

(8) *The non-basic F-loci of the j-th kind meet* W_p *in manifolds of dimension* $p-j$, *which are run over by a linear system of* ∞^{p-j-1} *involution curves. In the case of the linear non-basic F-loci these are involution curves attached to norm-curves.*

The particular cases, $j = p$, $j = p - 1$ are discussed in **4**.

The cases $j = -1$ and $j = -2$ differ from those just treated in that $(p-j)$-ics are plotted in a space of dimension respectively one or two greater than that of S_p. In these cases the $p+1$ coefficients $g_{p-j}(t_a), g_{p-j}(s_c)$ in **3** (E) are the $p+1$ coördinates of a point in S_{p-j} when this point is projected upon an S_p from the point $\pi_0 (j = -1)$, or line $\pi_1 (j = -2)$, in which the hyperosculating spaces t_a, s_c of N^{p-j} meet. Moreover the reference R_{p+1} in S_p to which these coördinates refer after the projection has vertices at t_a, s_c where S_p cuts N^{2p-1}. Hence

(9) *The generic curves,* $[W_p, S_p(t_1, \cdots, t_{j+2}, s_1, \cdots, s_{p-j-1})]$, *defined by the pencil* π *for* $j = -1, -2$ *are projections of the involution curve* $[I_{2p-j}^{(p-j)}, N^{p-j}]$ *of order* $\binom{2p-j-1}{p-j-1}$ *defined by* g_{p-j}*-ics of* π *with reference to* N^{p-j} *from the linear space* π_{-j-1} *in which the hyperosculating spaces of* N^{p-j} *with parameters* t_a, s_c *meet.*

We may therefore make the general statement that

(10) *The curves cut out on* W_p *by* $(p+1)$*-secant* S_p*'s of* N^{2p-1} *are either involution curves attached to norm-curves* [cf. (1), (7)], *or they are projections of such curves* [cf. (9)].

An entirely different aspect of these curves is brought out by the formula **3** (F) which we have not as yet used. This formula contains the coefficients of the factor f_p residual to g_{p-j} in the pencil π. Let N'^p be a norm-curve in S'_p with reference to which the p-ics f_p are plotted. Then $f_p(t_a), f_p(s_c)$ are

coördinates in S'_p with respect to the reference figure R'_{p+1} whose $p+1$ spaces hyperosculate N'^p at the points t_a, s_c of N'^p. As before $1/f_p(t_a)$, $1/f_p(s_c)$ are coördinates in S_p with respect to the reference figure R_{p+1} whose $p+1$ vertices are the points t_a, s_c of N^{2p-1}. Since f and $1/f$ are related by a Cremona transformation, we have the theorem:

(11) *Let the p-ics, f_p, of the pencil, $\pi \equiv \mu_{2p-j} - g^2{}_{p-j-1} \cdot \lambda_{j+2} \equiv -f_p \cdot g_{p-j}$, plotted with respect to the norm-curve N'^p in S'_p, define the involution curve $(I^{(p)}_{2p-j}, N'^p)$ of order $\binom{2p-j-1}{p-1}$. Let R'_{p+1} be the reference figure in S'_p whose S'_{p-1}'s hyperosculate N'^p with parameters t_a, s_c. Let S_p be the $(p+1)$-secant space of N^{2p-1} at the points R_{p+1} with parameters t_a, s_c. Then the regular Cremona transformation of order p with direct and inverse F-points at the points of R'_{p+1}, R_{p+1} respectively transforms the above involution curve into the generic curve of the section of W_p by S_p.*

A version of the inverse Cremona transformation is obtained by taking the canonizant of $(\alpha t)^{2p-1}$ in **3** (E). This canonizant, according to [**3** (1), (C)], is f_p itself. But the canonizant of **3** (E) is of degree p in the coefficients of g_{p-j}.

7. Applications to W_2, W_3, W_4. The generic section of W_2, the Weddle surface in S_3, by an S_2 on the points s_1, s_2, s_3 of the cubic curve N^3 on the six nodes $P_6{}^3$ of W_2 is a quartic curve which, according to the Morley-Conner theorem, generalized in **5** (1), contains ∞^1 configurations $(15_3, 20_4)$. These are the sections of the complete 6-points determined on N^3 by the pencil, $\pi = \mu_6 - f_3{}^2 = -f_2 f_4$, f_3 having roots s_1, s_2, s_3.

The general theorems of the preceding section, **6** (9) and **6** (11), present this curve under two new aspects. The first aspect is that of tetrads, f_4, of members of the pencil π. If these are plotted as points in S_4 with reference to an N^4, the generic sextic of the pencil determines 15 points of the involution curve $(I_6{}^{(4)}, N^4)$, of which 10 are in a particular osculating S_3 of N^4. For the particular sextic, $f_3{}^2$, these 15 points comprise three of type, $b_1 = s_2{}^2 s_3{}^2$; and three of type, $a_1 = s_1{}^2 s_2 s_3$, each counting four times. The latter three points, a_1, a_2, a_3, are on the line of intersection of the three osculating spaces s_1, s_2, s_3 of N^4; and they are double points of $(I_6{}^{(4)}, N^4)$. For, the osculating space s_1 contains a_1 counting four times, and a_2, a_3 each counting twice. But also the osculating plane $s_1{}^2$ contains a_1 counting four times, and a_1 is thus a node with tangents in the plane $s_1{}^2$. Hence $(I_6{}^{(4)}, N^4)$ of order 10, projected from the line on its three nodes a_i, yields the quartic plane section of W_2.

The second aspect is that of pairs f_2 of members of the pencil π. We take then in a plane π' a norm-conic N'^2 and the six-lines π circumscribed

about it, each six-line contributing 15 points on the involution curve $(I_6^{(2)}, N'^2)$ of order 5. The particular member, f_3^2, contributes a circumscribed triangle s_1, s_2, s_3 of N'^2, the 15 points being the three points $c_1 = s_1^2$, $c_2 = s_2^2$, $c_3 = s_3^2$ of contact; and the three vertices, $d_1 = s_2s_3$, $d_2 = s_1s_3$, $d_3 = s_1s_2$, each counting four times. The five points on the line s_3 of N'^2 are s_3^2 on N'^2 and d_1, d_2 each counting twice. Hence d_1, d_2, d_3 are nodes of the involution curve. According to **6** (11) the section of W_2 is the transform of this quintic $(I_6^{(2)}, N'^2)$ by the quadratic transformation A_{123} with F-points at its nodes d_1, d_2, d_3. The section is therefore a quartic curve. If t_1, t_2, t_3 is a triad of any member of π, the vertices of the two circumscribed triangles, t_1, t_2, t_3 and s_1, s_2, s_3, of N'^2 are on a conic. This conic is transformed by A_{123} into a line and thus we find on the section of W_2 the inscribed Morley-Conner configurations.

The conic N'^2 itself passes by A_{123} into the tri-cuspidal quartic curve whose envelope is the rational cubic of lines cut out on the plane of the section by the osculating planes of N^3. The cusp triangle cut out on the plane by N^3 is the triangle of inverse F-points. These interesting connections, and the particular cases which arise from sections of W_2 by planes on one or two nodes of W_2, are deserving of further study.

In passing to the consideration of W_3 in S_5 and W_4 in S_7 we utilize only the second aspect mentioned above, since we are interested primarily in the order of these loci. We consider the section of W_3 by the S_3 quadrisecant to N^5 at $f_4 = s_1, s_2, s_3, s_4$ and the associated pencil $\pi = \mu_8 - f_4^2 = -f_3 \cdot f_5$. The triads f_3 of members of the pencil are mapped by points f_3 in the space S'_3 of a cubic curve N'^3, which lie on the involution curve $(I_8^{(3)}, N'^3) = K^{21}$ of order 21. If $t_1, \cdots, t_8$ is a generic octavic of the pencil π, the osculating plane t_1 of N'^3 cuts K^{21} in the 21 points $t_1t_2t_3, \cdots, t_1t_7t_8$. The axis t_1t_2 of N'^3 cuts K^{21} in 6 points $t_1t_2t_3, \cdots, t_1t_2t_8$. The osculating plane t_1 of N'^3 is cut by the planes of N'^3 in the lines of a conic $K^2(t_1)$ which touches the tangent to N'^3 at the point t_1 of N'^3. The 7 axes cut out on the plane t_1 of N'^3 by planes $t_2, \cdots, t_7$ envelop $K^2(t_1)$ and their 21 meets are on K^{21}.

The pencil π has no other peculiarity than that it contains one square member, f_4^2. For this member the inscribed 8-plane of K^{21} collapses to a 4-plane. The seven axes enveloping $K^2(s_1)$ are now the tangent to N'^3 at s_1; and the three axes s_1s_2, s_1s_3, s_1s_4, each counting twice. The tangent, or axis s_1^2, meets these three axes in points $s_1^2s_2$, $s_1^2s_3$, $s_1^2s_4$. Thus these three points are contacts of a tritangent line of K^{21}, namely, the tangent to N'^3 at s_1. The 6 points of K^{21} on the axis s_1s_2 are now $s_1s_2s_3$, $s_1s_2s_4$, each counting twice; and $s_1^2s_2$ and $s_1s_2^2$ (the contact of the tangent s_2 of $K^2(s_1)$), each counting once. The plane s_1 cuts K^{21} in points $d_2 = s_1s_3s_4$, $d_3 = s_1s_2s_4$, $d_4 = s_1s_2s_3$, each

counting four times; the points $s_1^2s_2$, $s_1^2s_3$, $s_1^2s_4$, each counting twice; and the points $s_1s_2^2$, $s_1s_3^2$, $s_1s_4^2$, each counting once. Since the point $d_1 = s_1s_2s_3$ is four-fold on each of the three planes s_1, s_2, s_3 containing it, it is a four-fold point of K^{21}. Thus the vertices d_i of the tetrahedron formed by the planes s_i of N'^3 are four-fold on K^{21}, and the edges s_is_j of this tetrahedron meet K^{21} in the two further points $s_i^2s_j$, $s_is_j^2$.

According to **6** (11) the section of W_3 by the $S_3(s_1 \cdots s_4)$ is the transform of K^{21} by the regular cubic transformation A_{1234} with F-points at $d_1, \cdots, d_4$ on N'^3 and with inverse F-points at $s_1, \cdots, s_4$ on N^5. Since the order of the transform of a curve by A_{1234} is reduced by two for each branch through an F-point, and by one for each crossing of an F-line joining two F-points, the transform of K^{21} by A_{1234} has the order $3 \cdot 21 - 2 \cdot 4 \cdot 4 - 1 \cdot 6 \cdot 2 = 19$. This transform L^{19} has triple points at $s_1, \cdots, s_4$. Due to the contacts of K^{21} with the tangent s_1^2 in S'_3 the tangents of L^{19} at the triple point s_1 are the three lines s_1s_2, s_1s_3, s_1s_4. Hence, making use also of **5** (1),

(1) *The section of W_3 in S_5 by a quadri-secant $S_3(s_1 \cdots s_4)$ of N^5 is a curve of order* 19 *with triple points at the four points s_i on N^5 and tangents s_is_j at the triple point s_i. This curve contains ∞^1 inscribed configurations, each consisting of* 56 *points,* 70 *lines, and* 56 *planes with the following incidences: each point is on* 5 *lines and* 10 *planes, each plane is on* 5 *lines and* 10 *points, and each line is on* 4 *points,* 16 *lines, and* 4 *planes.*

The inverse transformation, A^{-1}_{1234}, transforms L^{19} back into K^{21}, the reduction being $3 \cdot 19 - 4 \cdot 3 \cdot 2 - 12 \cdot 1 = 21$, where the reduction $12 \cdot 1$ arises from the contacts of the edges of the tetrahedron s_i at the triple points. We have thus the confirmation of an earlier result [cf. [1], (81)], and the more precise information given by

(2) *The W_3 in S_5 has the order* 19 *and has the triple curve N^5 on P_8^5 and therefore also triple lines p_ip_j. The tangent cone at a point s of N^5 contains the bisecants of N^5 on s.*

This explains the behavior of a trisecant plane of N^5 which must cut W_3 in 19 points. Through any point of W_3 there is just one such trisecant plane, namely, the plane r_1, r_2, r_3 of **2** (17b). This plane cuts W_3 in the four ordinary points obtained by the variation of the signs of z_{r_i}. The intersections with the triple curve N^5 at the points r_i account for 9, and the contacts of the plane with W_3 along the line r_ir_j at r_i and r_j account for the remaining 6, of the 19 intersections. It is these four ordinary points in the trisecant plane $s_2s_3s_4$ which pass by A^{-1}_{1234} into the four-fold point of K^{21} at the point d_1.

We examine next the intersection of W_3 by the quadrisecant S_3 of N^5 on s_1, s_2, s_3 roots of $g_3 = 0$, and t_1, a point of $P_8{}^5$. Since the cone of lines on t_1 to points of N^5 is an F-locus of the third kind lying on W_3, the three lines from t_1 to s_1, s_2, s_3 will separate from the intersection leaving a curve L^{16}, which is associated with the pencil $\pi = \mu_7 - g_3{}^2\lambda_1 = -f_3 \cdot g_4$. We first examine the involution curve, $(I_7{}^{(3)}, N'^3) = K^{15}$, determined by the triads f_3 of the pencil with reference to an N'^3 in S'_3. The four planes s_1, s_2, s_3, t_1 of N'^3 form a tetrahedron with respective opposite vertices d_1, d_2, d_3, d_4. By considering the multiplicities of the 15 points of K^{15} on each of the four planes, and of the 5 points of K^{15} on each of the six edges, of this tetrahedron, we find that: (a) the point d_4 is four-fold, and the points d_i are double, on K^{15}; (b) the point $s_i{}^2s_j$ is on K^{15}, and the tangent to K^{15} at the point is in the plane s_i; (c) the point $s_i{}^2t_1$ on the edge s_it_1 is on K^{15} with tangent neither in the plane s_i nor in the plane t_1; and (d) the pairs of tangents of K^{15} at the points d_i are in the plane t_1 $[i, j = 1, 2, 3]$. The transformation A_{1234} with F-points at d_1, d_2, d_3, d_4 and respective inverse F-points at s_1, s_2, s_3, t_1 in the S_3-section of W_3 transforms K^{15} into L^{16} with multiplicities 2 at s_i and 6 at t_1, due to the 4 F-points at $d_1, \cdots, d_4$ and the 9 F-points on the edges. Due to (b), the tangents of L^{16} at the node s_i are s_is_j and s_is_k. Due to (c), the line s_1t_1 on W_3 cuts L^{16} at a point distinct from s_1 and t_1. Due to (d), the tangents of L^{16} at t_1 are found two in each of the three planes $t_1s_is_j$. We have thus confirmed the 9-fold character of the F-point p_1 of W_3 [cf. [1], (81)]. The trisecant plane $t_1s_1s_2$ cuts L^{16} in 8 points at t_1, in 3 points at s_1 and at s_2, and in one point on each of t_1s_1 and t_1s_2. These points all are on F-loci of the second or third kinds. Thus the F-locus of the first kind, $\pi^{(1)}_{2\ldots 8}$, the locus of S_2's on t_1 and two variable points of N^5 meets W_3 in no significant locus. It is indeed paired with $\pi_1{}^{(1)}$, the directions about t_1, and thus the two F-loci meet each other and W_3 only in the directions about t_1 on W_3, apart from intersections on F-loci of higher kinds.

The intersection of W_3 by the quadrisecant S_3 of N^5 on s_1, s_2, roots of $g_2 = 0$, and on t_1, t_2, points of $P_8{}^5$, is associated with the pencil, $\pi = \mu_6 - g_2{}^2\lambda_2 = -f_3 \cdot g_3$. This intersection must contain as a part the four bisecants t_is_j, and it must contain the F-line t_1t_2 as a five-fold intersection, three-fold due to the multiplicity of t_1t_2 and twice as a contact, this being due to the order of the residual intersection L^{10} associated with the triads g_3 of the pencil [cf. **6** (4)]. The triads f_3 of this pencil determine in S'_3 with reference to N'^3 an involution curve, $(I_6{}^{(3)}, N'^3) = K^{10}$. The four planes of N'^3 with parameters t_1, t_2, s_1, s_2 have respective opposite vertices d_1, d_2, e_1, e_2. The special sextic $g_2{}^2\lambda_2$ requires that K^{10} have the following properties: (a) the points

d_1, d_2 are nodes, e_1, e_2 simple points, of K^{10}, the edge e_1e_2 being tangent at e_1, e_2; (b) the edge s_1s_2 meets K^{10} in two simple points $s_1{}^2s_2$, $s_1s_2{}^2$, the tangent at $s_1{}^2s_2$ lying in the plane s_1; and (c) the edge s_it_j meets K^{10} in the point $s_i{}^2t_j$. The transformation A_{1234} with F-points at d_1, d_2, e_1, e_2 and respective inverse F-points at t_1, t_2, s_1, s_2 on N^5 transforms K^{10} into L^{10} with multiplicities $2, 2, 1, 1$ at t_1, t_2, s_1, s_2, due to the multiplicities of K^{10} at the four F-points and at the eight F-points of the second kind on the edges. Due to (b), the edge s_1s_2 touches L^{10} at s_1 and s_2; and, due to (c), L^{10} is crossed by s_it_j. Due also to the contacts at e_1, e_2, the tangents to L^{10} at the points on the edge t_1t_2 are respectively on the planes containing this edge. Thus L^{10} and K^{10} are each related to their respective tetrahedra in precisely the same way as might be inferred directly from the mutual relation of f_3, g_3 to π. The most striking new property that has appeared is:

(3) *A quadrisecant S_3 of N^5 with two crossings at t_1, t_2 has a contact of second order with W_3 along the line t_1t_2 in addition to the expected triple intersection.*

We examine finally the intersection of W_3 by the quadrisecant S_3 of N^5 on s_1 and on t_1, t_2, t_3, points of $P_8{}^5$. This is made up of the plane $t_1t_2t_3$ and a residual curve part of which is made up of t_is_1 and of the lines t_it_j to a certain multiplicity. The significant part of the curve is associated with the pencil $\pi = \mu_5 - g_1{}^2\lambda_3 = -f_3 \cdot g_2$. In this case we observe at once that the duads g_2 of the pencil determine a Lüroth quartic curve in the plane S_2 cut out on S_3 by the $S_4(p_4, \cdots, p_8) = \pi^{(1)}_{123}$ [cf. **6** (6)]. Thus

(4) *The 2-way of order* 9 [cf. [1], p. 489] *cut out on W_3 by the F-locus $\pi^{(1)}_{123}$, the ten planes $p_4p_5p_6, \cdots, p_6p_7p_8$ being disregarded, contains a linear system of planar Lüroth quartics.*

We conclude with a discussion of the intersection of W_4 in S_7 with an S_4 5-secant to N^7 at points $s_1, \cdots, s_5$ of N^7 given by $f_5 = 0$. In the case of W_4, the bisecant locus of N^7 is an F-locus of the fourth kind contained simply on W_4, whence the intersection in question will contain as a part the 10 bisecant lines of N^7 joining the points s. The remaining significant intersection is associated with the pencil, $\pi \equiv \mu_{10} - f_5{}^2 = -f_4 \cdot f_6$. We plot the tetrads f_4 of members of this pencil in S'_4 with reference to an N'^4 to obtain the involution curve $(I^{(4)}_{10}, N'^4) = K^{84}$. The degenerate member $f_5{}^2$ contributes five S_3's of N'^4 with parameters $s_1, \cdots, s_5$ whose five vertices are $d_i = s_js_ks_ls_m$. An examination of the 84 intersections of these five S_3's with K^{84} yields the following results: (a) the five points d_i are 8-fold on K^{84};

(b) the edge $d_i d_j$ touches K^{84} at three points $d_{ij,k} = s_k^2 s_l s_m$; and (c) the plane $d_i d_j d_k$ cuts K^{84} at $d_{ijk} = s_l^2 s_m^2$. The quartic transformation A_{12345} with F-points at $d_1, \cdots, d_5$, and inverse F-points at the points $s_1, \cdots, s_5$ on N^7 in S_7 transforms K^{84} into the section L of W_4 by the 5-secant S_4. Due to the 5 8-fold F-points d_i, the 30 repeated F-points on edges $d_i d_j$, and the 10 F-points on planes $d_i d_j d_k$, the order of the transform is $4 \cdot 84 - 5 \cdot 8 \cdot 3 - 30 \cdot 2 \cdot 2 - 10 \cdot 1 = 86$, and the section L^{86} has 12-fold points at the five points s_i. On adding the 10 lines of this five-point we have the theorem:

(5) *W_4 in S_7 has the order* 96, *and it contains N^7 and the* 45 *lines $p_i p_j$ as* 16-*fold curves.*

The point $d_{ij,k}$ on K^{84} by virtue of (b) passes into a direction on L^{86} on the trisecant plane $s_k s_l s_m$ of N^7 at s_k, this direction counting doubly. The six trisecant planes on s_i thus account for the multiplicity 12 of L^{86} at s_i. The point d_{ijk} on K^{84}, by virtue of (c), passes into a point of L^{86} on the line $s_1 s_2$. Thus L^{86} is transformed back into K^{84} since $4 \cdot 86 - 5 \cdot 12 \cdot 3 - 10 \cdot 2 - 30 \cdot 2 \cdot 1 = 84$. The 8-fold point of K^{84} at d_1 arises from the eight generic points of W_4 in the quadric-secant S_3 on $s_2, \cdots, s_5$ [cf. **2** (17c)]. The remaining 78 intersections of this S_3 with L^{86} are accounted for by the 48 at $s_2, \cdots, s_5$, by the 6 on the edges $s_2 s_3, \cdots$, and by the $4 \cdot 3 \cdot 2$ directions at s_2 in the planes $s_2 s_3 s_4, \cdots$.

REFERENCES.

[1] A. B. Coble, "A generalization of the Weddle Surface, . . . ," *American Journal of Mathematics*, vol. 52 (1930), pp. 439-500.

[2] A. B. Coble, "Hyperelliptic functions and irrational binary invariants I," *American Journal of Mathematics*, vol. 54 (1932), pp. 425-452.

[3] A. B. Coble, "Algebraic geometry and theta functions," *Colloquium Lectures of the American Mathematical Society*, vol. 10, New York (1929).

[4] F. Morley and J. R. Conner, "Plane sections of the Weddle surface," *American Journal of Mathematics*, vol. 31 (1909), pp. 263-270.

[5] A. B. Coble, "The geometry of the Weddle manifold, W_p," *Bulletin of the American Mathematical Society*, vol. 41 (1935), April number.

URBANA, ILLINOIS.

A THEORY OF POSITIVE INTEGERS IN FORMAL LOGIC.*
PART II.

By S. C. KLEENE.

15. Formal definition: initial values, induction. If L is an intuitive function which associates well-formed expressions $L(x_1, \cdots, x_n)$ with n-tuples $(x_1, \cdots, x_n)$ of well-formed expressions, then L shall be said to be *defined (formally)* by $\mathbf{L}$ if $\mathbf{L}(x_1, \cdots, x_n)$ conv $L(x_1, \cdots, x_n)$ for each set $(x_1, \cdots, x_n)$ for which L is defined. By the "definition" of a function which correlates intuitive mathematical objects, we shall mean the definition of the function which correlates the corresponding well-formed formulas, in case corresponding formulas have been designated. By the "definition" of a sequence $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \cdots$, we shall mean the definition of a function L whose values for the arguments $1, 2, 3, \cdots$ are $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \cdots$, respectively. That is, $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \cdots$ shall be defined (formally) by $\mathbf{L}$, if $\mathbf{L}(\mathbf{i})$ conv $\mathbf{A}_i$ $(i = 1, 2, 3, \cdots)$.

Closely connected with the formal theory of this paper, there is an intuitive theory concerning the formal definition of the functions involved. For the preceding sections, this may be summarized by the following theorem, each part of which can be established, either directly, with the aid of the first, or by means of considerations used above in formal proofs.†

15I. *Suppose that x and y are given positive integers of intuitive logic.* a. $\mathbf{x}$ *conv* $\lambda fa \cdot f(\cdots x$ times $\cdots f(a) \cdots)$. b. *If* $x + y = z$, $\mathbf{x} + \mathbf{y}$ *conv* $\mathbf{z}$. c. *If* $xy = z$, $\mathbf{xy}$ *conv* $\mathbf{z}$. d. $\mathbf{F}^{\mathbf{x}}(\mathbf{A})$ *conv* $\mathbf{F}(\cdots x$ times $\cdots \mathbf{F}(\mathbf{A}) \cdots)$. e. $I^{\mathbf{x}}$ *conv* I; $I(\mathbf{A})$ *conv* $\mathbf{A}$. f. *If* $x^y = z$, $\mathbf{x}^{\mathbf{y}}$ *conv* $\mathbf{z}$. g. *If* $x > y$ *and* $x - y = z$, $\mathbf{x} - \mathbf{y}$ *conv* $\mathbf{z}$; *if* $x \leq y$, $\mathbf{x} - \mathbf{y}$ *conv* 1. h. *If* $x \leq y$, $\min(\mathbf{x}, \mathbf{y})$ *conv* $\min(\mathbf{y}, \mathbf{x})$ *conv* $\mathbf{x}$. i. *If* $x \geq y$, $\max(\mathbf{x}, \mathbf{y})$ *conv* $\max(\mathbf{y}, \mathbf{x})$ *conv* $\mathbf{x}$. j. $1 \circ 1$ *conv* $1 \circ 2$ *conv* $2 \circ 1$ *conv* 1; $2 \circ 2$ *conv* 2. k. *If* $x \leq y$, $\epsilon^{\mathbf{x}}_{\mathbf{y}}$ *conv* 1; *if* $x > y$, $\epsilon^{\mathbf{x}}_{\mathbf{y}}$ *conv* 2. l. *If* $x \neq y$, $\delta^{\mathbf{x}}_{\mathbf{y}}$ conv 1; *if* $x = y$, $\delta^{\mathbf{x}}_{\mathbf{y}}$ *conv* 2.‡

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† 15I is stated with the aid of the convention that if n represents a positive integer of intuitive logic, then $\mathbf{n}$ shall represent the corresponding positive integer $S(\cdots n-1$ times $\cdots S(1) \cdots)$ of our formal theory.

‡ This theorem includes the assertion that the intuitive functions $x + y$, xy, x^y, $x - y$, $\min(x, y)$, $\max(x, y)$ are definable (for positive integral arguments and values). Also, constant and identity functions of positive integers are definable: *If* $n, x_1, \cdots, x_n$ *are given positive integers, then* $\mathfrak{C}(\mathbf{n}, \mathbf{A}, \mathbf{x}_1, \cdots, \mathbf{x}_n)$ *conv* $\mathbf{A}$ *and* $\mathfrak{T}_{ni}(\mathfrak{T}_n(\mathbf{x}_1, \cdots, \mathbf{x}_n))$ *conv* $\mathbf{x}_i$, *where* $\mathfrak{T}_n \rightarrow \lambda\rho_1 \cdots \rho_n f_1 \cdots f_n a \cdot f_1^{\rho_1}(\cdots f_n^{\rho_n}(a))$ *and* $\mathfrak{T}_{ni} \rightarrow \lambda\rho f \cdot \rho(I, \cdots i-1$ times $\cdots, I, f, I, \cdots n-i$ times $\cdots, I)$ (cf. §§ 7, 8).

The remainder of this paper is devoted to further developments of the theories of formal definition and of formal proof in conjunction with each other.

15II. *A necessary condition that a function of positive integers, the values of which are well-formed expressions, be definable is that all the values have the same free symbols.*

This is a consequence of C5VI.

15III(k). *If* $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k, \boldsymbol{F}$ *have the same free symbols, then the sequence* $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k, \boldsymbol{F}(1), \boldsymbol{F}(2), \cdots$ *is definable by a formula* $\mathbf{L}$ *such that* $N(\boldsymbol{X}) \vdash' \boldsymbol{L}(\boldsymbol{k} + \boldsymbol{X}) =' \boldsymbol{F}(\boldsymbol{X})$.*

Proof. If $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same free symbols, then, by C7I, there exists a formula $\boldsymbol{B}$ such that $\boldsymbol{B}(1)$ conv $\lambda \boldsymbol{n} \cdot I^{\boldsymbol{n}}(\boldsymbol{A})$ and $\boldsymbol{B}(2)$ conv $\boldsymbol{F}$. Let $\boldsymbol{L} \rightarrow \lambda \boldsymbol{n} \cdot \boldsymbol{B}(\min(2, \boldsymbol{n}), \boldsymbol{n} - 1)$.† Then it is clear from 15Ie, g, h that $\boldsymbol{L}$ defines $\boldsymbol{A}, \boldsymbol{F}(1), \boldsymbol{F}(2), \cdots$. Also $N(\boldsymbol{X}) \vdash' \boldsymbol{L}(1 + \boldsymbol{X}) =' \boldsymbol{F}(\boldsymbol{X})$, since, assuming $N(\boldsymbol{X})$, we have $\boldsymbol{L}(1 + \boldsymbol{X})$ conv $\boldsymbol{B}(\min(2, S(\boldsymbol{X})), S(\boldsymbol{X}) - 1)$, $= \boldsymbol{B}(2, \boldsymbol{X})$ (13.2, 12.4, 11.2), conv $\boldsymbol{F}(\boldsymbol{X})$. Thus 15III(1) is established. Moreover 15III($k+1$) is a consequence of 15III(k) and 15III(1).‡ Thus 15III(k) is established by an intuitive induction with respect to k.

COROLLARY. *If* $\boldsymbol{A}_{i_1 \ldots i_n}$ $(i_1, \cdots, i_n = 1, \cdots, k)$ *have the same free symbols, then a formula* $\mathbf{L}$ *can be found such that* $\mathbf{L}(\boldsymbol{i}_1, \cdots, \boldsymbol{i}_n)$ *conv* $\boldsymbol{A}_{i_1 \ldots i_n}$.

This follows from 15III by induction with respect to n, since, given the hypothesis with $n+1$ replacing n, we can, by using the corollary as stated, find k formulas $\boldsymbol{L}_{i_1}$ such that $\boldsymbol{L}_{i_1}(\boldsymbol{i}_2, \cdots, \boldsymbol{i}_{n+1})$ conv $\boldsymbol{A}_{i_1 \ldots i_{n+1}}$, and then by 15III find an $\boldsymbol{L}$ such that $\boldsymbol{L}(\boldsymbol{i}_1)$ conv $\boldsymbol{L}_{i_1}$.

15IV. *If the free symbols of* $\boldsymbol{F}$ *are included among those of* $\boldsymbol{A}$, *then the sequence* $\boldsymbol{A}, \boldsymbol{F}(\boldsymbol{A}), \boldsymbol{F}(\boldsymbol{F}(\boldsymbol{A})), \cdots$ *is definable by a formula* $\mathbf{L}$ *such that* $N(\boldsymbol{X}) \vdash' \boldsymbol{L}(S(\boldsymbol{X})) =' \boldsymbol{F}(\boldsymbol{L}(\boldsymbol{X}))$.

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* For the notation $\vdash' ='$ see the last paragraph of § 2.

† When a heavy-typed letter represents occurrences of a proper symbol in a formula, we shall suppose the symbol to be one whose only occurrences in the formula are those represented by the occurrences of the letter, unless the contrary is implied by the conventions (1) and (2) of § C3. Thus $\boldsymbol{n}$ is here supposed to be distinct from the proper symbols of $\boldsymbol{A}$ and $\boldsymbol{B}$, but in "$\lambda \boldsymbol{n} \cdot \boldsymbol{M}$" $\boldsymbol{n}$ must occur in $\boldsymbol{M}$ as a free symbol in order that $\boldsymbol{M}$ and $\lambda \boldsymbol{n} \cdot \boldsymbol{M}$ be well-formed.

‡ Using the fact that if $\boldsymbol{L}'$ defines $\boldsymbol{A}_2, \cdots, \boldsymbol{A}_{k+1}, \boldsymbol{F}(1), \boldsymbol{F}(2), \cdots$, then $\boldsymbol{L}'$ has the same free symbols as $\boldsymbol{A}_2, \cdots, \boldsymbol{A}_{k+1}$ and $\boldsymbol{F}$ (cf. C5VI). Similarly below.

Proof. By 15III, the sequence $\boldsymbol{A}, \boldsymbol{F}^1(\boldsymbol{A}), \boldsymbol{F}^2(\boldsymbol{A}), \cdots$ is definable by an $\boldsymbol{L}$ such that $N(\boldsymbol{Y}) \vdash' \boldsymbol{L}(S(\boldsymbol{Y})) =' \boldsymbol{F}^{\boldsymbol{Y}}(\boldsymbol{A})$. By 15Id, $\boldsymbol{L}$ defines $\boldsymbol{A}, \boldsymbol{F}(\boldsymbol{A}), \boldsymbol{F}(\boldsymbol{F}(\boldsymbol{A})), \cdots$. Assume $N(\boldsymbol{X})$. Case 1: $\epsilon_1{}^{\boldsymbol{X}} = 1$. Then $\boldsymbol{X} = 1$ (14.8, 12.17, 12.8). Hence $\boldsymbol{L}(S(\boldsymbol{X})) = \boldsymbol{L}(S(1))$, conv $\boldsymbol{F}(\boldsymbol{A})$ (since $\boldsymbol{L}$ defines $\boldsymbol{A}, \boldsymbol{F}(\boldsymbol{A}), \cdots$), conv $\boldsymbol{F}(\boldsymbol{L}(1))$, $= \boldsymbol{F}(\boldsymbol{L}(\boldsymbol{X}))$. Case 2: $\epsilon_1{}^{\boldsymbol{X}} = 2$. Then $\boldsymbol{X} > 1$ (14.9), and $\boldsymbol{X} = S(\boldsymbol{X} - 1)$ (12.5). Hence $\boldsymbol{L}(S(\boldsymbol{X})) = \boldsymbol{L}(S(S(\boldsymbol{X} - 1)))$, $= \boldsymbol{F}^{S(\boldsymbol{X}-1)}(\boldsymbol{A})$ (since $N(\boldsymbol{Y}) \vdash' \boldsymbol{L}(S(\boldsymbol{Y})) =' \boldsymbol{F}^{\boldsymbol{Y}}(\boldsymbol{A})$), conv $\boldsymbol{F}(\boldsymbol{F}^{\boldsymbol{X}-1}(\boldsymbol{A}))$, $= \boldsymbol{F}(\boldsymbol{L}(S(\boldsymbol{X} - 1)))$, $= \boldsymbol{F}(\boldsymbol{L}(\boldsymbol{X}))$. Hence, by cases (C9I), $\boldsymbol{L}(S(\boldsymbol{X})) =' \boldsymbol{F}(\boldsymbol{L}(\boldsymbol{X}))$.*

15V. *If the free symbols of* $\boldsymbol{F}$ *are included among those of* $\boldsymbol{A}$, *then the sequence* $\boldsymbol{A}, \boldsymbol{F}(1, \boldsymbol{A}), \boldsymbol{F}(2, \boldsymbol{F}(1, \boldsymbol{A})), \cdots$ *can be defined by a formula* $\boldsymbol{L}$ *such that* $N(\boldsymbol{X}) \vdash' \boldsymbol{L}(S(\boldsymbol{X})) =' \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{L}(\boldsymbol{X}))$.

Proof. Let $\boldsymbol{A}' \to \lambda \boldsymbol{x} \cdot I^{\boldsymbol{x}}(\boldsymbol{A})$ and $\boldsymbol{F}' \to \lambda \boldsymbol{p}\boldsymbol{x} \cdot \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{p}(\boldsymbol{x} - 1))$. By 15IV, $\boldsymbol{A}', \boldsymbol{F}'(\boldsymbol{A}'), \boldsymbol{F}'(\boldsymbol{F}'(\boldsymbol{A}')), \cdots$ is definable by a formula $\boldsymbol{L}'$ such that $N(\boldsymbol{X}) \vdash' \boldsymbol{L}'(S(\boldsymbol{X})) =' \boldsymbol{F}'(\boldsymbol{L}'(\boldsymbol{X}))$. Let $\boldsymbol{L} \to \lambda \boldsymbol{x} \cdot \boldsymbol{L}'(\boldsymbol{x}, \boldsymbol{x} - 1)$. Then, assuming $N(\boldsymbol{X})$, $\boldsymbol{L}(S(\boldsymbol{X}))$ conv $\boldsymbol{L}'(S(\boldsymbol{X}), S(\boldsymbol{X}) - 1)$, $= \boldsymbol{L}'(S(\boldsymbol{X}), \boldsymbol{X})$ (11.2), $= \boldsymbol{F}'(\boldsymbol{L}'(\boldsymbol{X}), \boldsymbol{X})$, conv $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{L}'(\boldsymbol{X}, \boldsymbol{X} - 1))$, conv $\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{L}(\boldsymbol{X}))$. Thus $N(\boldsymbol{X}) \vdash' \boldsymbol{L}(S(\boldsymbol{X})) =' \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{L}(\boldsymbol{X}))$. Similarly, using 15Ig, $\boldsymbol{L}(S(\boldsymbol{i}))$ conv $\boldsymbol{F}(\boldsymbol{i}, \boldsymbol{L}(\boldsymbol{i}))$ $(i = 1, 2, \cdots)$. Also $\boldsymbol{L}(1)$ conv $\boldsymbol{L}'(1, 1 - 1)$, conv $\boldsymbol{L}'(1, 1)$, conv $\boldsymbol{A}'(1)$, conv $I^1(\boldsymbol{A})$, conv $\boldsymbol{A}$. Hence, by intuitive induction with respect to i, $\boldsymbol{L}$ defines $\boldsymbol{A}, \boldsymbol{F}(1, \boldsymbol{A}), \cdots$.

15VI(k). *If the free symbols of* $\boldsymbol{F}$ *are included among those of* $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k$, *and* $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k$ *have the same free symbols, then the sequence* $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k, \boldsymbol{F}(1, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_k), \boldsymbol{F}(2, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_{k+1}), \cdots$, *where* $\boldsymbol{A}_i$ *denotes the i-th member of the sequence, is definable.*

Proof. For $k = 1$, this is included in 15V.

Suppose k a given positive integer ≥ 2, and let

$$\mathfrak{A}_{k1} \to \lambda f a_1 \cdots a_k \cdot f(1, a_1, \cdots, a_k),$$
$$\mathfrak{A}_{k2} \to \lambda f a_1 \cdots a_k \cdot f(2, a_2, \cdots, a_k, \mathfrak{A}_{k1}(f, a_1, \cdots, a_k)),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathfrak{A}_{kk} \to \lambda f a_1 \cdots a_k \cdot f(\boldsymbol{k}, a_k, \mathfrak{A}_{k1}(f, a_1, \cdots, a_k), \cdots, \mathfrak{A}_{k,k-1}(f, a_1, \cdots, a_k)),$$
$$\mathfrak{D}_{k1} \to \lambda \rho \cdot \rho(\lambda t_1 \cdots t_k \cdot t_2(I, \cdots k+1 \text{ times} \cdots, I, t_3(I, \cdots, I, \cdots,$$
$$t_k(I, \cdots, I, t_1) \cdots))),$$

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* In C9I, $\vdash \boldsymbol{C}$ may be replaced by $\vdash' \boldsymbol{M} =' \boldsymbol{N}$, since C9I may be applied with $\boldsymbol{T}(\boldsymbol{N})$ taken as $\boldsymbol{C}$ and $\vdash$ replaced by $\boldsymbol{T}(\boldsymbol{M}) \vdash$, and with $\boldsymbol{T}(\boldsymbol{M})$ taken as $\boldsymbol{C}$ and $\vdash$ replaced by $\boldsymbol{T}(\boldsymbol{N}) \vdash$.

$$\mathfrak{D}_{k2} \to \lambda\rho \cdot \rho(\lambda t_1 \cdots t_k \cdot t_1(I, \cdots, I, t_3(I, \cdots, I, \cdots, t_k(I, \cdots, I, t_2) \cdots))),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathfrak{D}_{kk} \to \lambda\rho \cdot \rho(\lambda t_1 \cdots t_k \cdot t_1(I, \cdots, I, t_2(I, \cdots, I, \cdots, t_{k-1}(I, \cdots, I, t_k) \cdots))),$$
$$\mathfrak{F}_k \to \lambda\sigma\rho\mu \cdot \mu(\mathfrak{D}_{k2}(\rho), \cdots, \mathfrak{D}_{kk}(\rho), \lambda f a_1 \cdots a_k \cdot f(\boldsymbol{k} + \sigma, \mathfrak{D}_{k1}(\rho, f, a_1, \cdots, a_k), \cdots, \mathfrak{D}_{kk}(\rho, f, a_1, \cdots, a_k))),$$
$$\mathfrak{A}_{k,k+i} \to \lambda f a_1 \cdots a_k \cdot f(\boldsymbol{k} + \boldsymbol{i}, \mathfrak{A}_{ki}(f, a_1, \cdots, a_k), \cdots, \mathfrak{A}_{k,i+k-1}(f, a_1, \cdots, a_k)),$$
$$\mathfrak{B}_{ki} \to \lambda\mu \cdot \mu(\mathfrak{A}_{ki}, \cdots, \mathfrak{A}_{k,i+k-1}) \qquad (i = 1, 2, \cdots).$$

By 15V, there exists a $\mathfrak{M}_k$ which defines $\mathfrak{F}_k(1, \mathfrak{B}_{k1}), \mathfrak{F}_k(2, \mathfrak{F}_k(1, \mathfrak{B}_{k1})), \cdots$.* We observe for $i = 1, \cdots, k$ successively that $\mathfrak{A}_{ki}(I, \cdots k+1$ times $\cdots, I)$ conv I. Assuming that $\mathfrak{A}_{ki}(I, \cdots, I)$ conv I for $i = p, \cdots, p + k - 1$, it follows that $\mathfrak{A}_{k,p+k}(I, \cdots, I)$ conv I. Hence, by intuitive induction, $\mathfrak{A}_{ki}(I, \cdots, I)$ conv I $(i = 1, 2, \cdots)$. Consequently $\mathfrak{D}_{kq}(\mathfrak{B}_{ki})$ conv $\mathfrak{A}_{k,i+q-1}$ $(i = 1, 2, \cdots;\ q = 1, \cdots, k)$. It follows that $\mathfrak{F}_k(\boldsymbol{i}, \mathfrak{B}_{ki})$ conv $\mathfrak{B}_{k,i+1}$ $(i = 1, 2, \cdots)$. Hence $\mathfrak{M}_k$ defines $\mathfrak{B}_{k2}, \mathfrak{B}_{k3}, \cdots$. By 15III, there exists an $\mathfrak{N}_k$ which defines $\mathfrak{A}_{k1}, \cdots, \mathfrak{A}_{kk}, \mathfrak{D}_{kk}(\mathfrak{M}_k(1)), \mathfrak{D}_{kk}(\mathfrak{M}_k(2)), \cdots$. Since $\mathfrak{D}_{kk}(\mathfrak{B}_{ki})$ conv $\mathfrak{A}_{k,i+k-1}$, $\mathfrak{N}_k$ defines $\mathfrak{A}_{k1}, \cdots, \mathfrak{A}_{kk}, \mathfrak{A}_{k,k+1}, \cdots$. By 15III, there exists an $\boldsymbol{L}$ which defines the sequence $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k, \mathfrak{N}_k(1, \boldsymbol{F}, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_k), \mathfrak{N}_k(2, \boldsymbol{F}, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_k), \cdots$, and hence the sequence $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_k, \mathfrak{A}_{k1}(\boldsymbol{F}, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_k), \mathfrak{A}_{k2}(\boldsymbol{F}, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_k), \cdots$. The $\mathfrak{A}_{ki}$ were so chosen that $\mathfrak{A}_{ki}(\boldsymbol{F}, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_k)$ conv $\boldsymbol{A}_{k+i}$. Hence $\boldsymbol{L}(\boldsymbol{i})$ conv $\boldsymbol{A}_i$ $(i = 1, 2, \cdots)$.

15VII. *If the free symbols of* $\boldsymbol{F}$ *are included among those of* $\boldsymbol{A}_1$, *then the sequence* $\boldsymbol{A}_1, \boldsymbol{F}(1, \boldsymbol{A}_1), \boldsymbol{F}(2, \boldsymbol{A}_2, \boldsymbol{A}_1), \boldsymbol{F}(3, \boldsymbol{A}_3, \boldsymbol{A}_2, \boldsymbol{A}_1), \cdots$, *where* $\boldsymbol{A}_i$ *denotes the i-th member of the sequence, is definable.*

Proof. Let $\mathfrak{B} \to \lambda\mu f a \cdot \mu(2, f(1, a), a)$ and $\mathfrak{W} \to \lambda\sigma\rho\mu f a \cdot \rho(\lambda\tau \cdot I^\tau(\mu(S^2(\sigma), \rho(f, f, a))), f, a)$. By 15V, there exists a formula $\mathfrak{K}$ which defines $\mathfrak{B}, \mathfrak{W}(1, \mathfrak{B}), \mathfrak{W}(2, \mathfrak{W}(1, \mathfrak{B})), \cdots$. $\mathfrak{K}(2)$ conv $\mathfrak{W}(1, \mathfrak{B})$ (by the def. of $\mathfrak{K}$), conv $\lambda\mu f a \cdot \mathfrak{B}(\lambda\tau \cdot I^\tau(\mu(S^2(1), \mathfrak{B}(f, f, a))), f, a)$ (using the def. of $\mathfrak{W}$), conv $\lambda\mu f a \cdot \{\lambda\tau \cdot I^\tau(\mu(S^2(1), \mathfrak{B}(f, f, a)))\}(2, f(1, a), a)$ (using the def. of $\mathfrak{B}$), conv $\lambda\mu f a \cdot \mu(3, \mathfrak{B}(f, f, a), f(1, a), a)$, conv $\lambda\mu f a \cdot \mu(3, \mathfrak{K}(1, f, f, a), f(1, a), a)$ (by the def. of $\mathfrak{K}$). Assume $\mathfrak{K}(\boldsymbol{i} + 1)$ conv $\lambda\mu f a \cdot \mu(\boldsymbol{i} + 2, \mathfrak{K}(\boldsymbol{i}, f, f, a), \cdots, \mathfrak{K}(1, f, f, a), f(1, a), a)$. Then $\mathfrak{K}(\boldsymbol{i} + 2)$ conv $\mathfrak{W}(\boldsymbol{i} + 1, \mathfrak{K}(\boldsymbol{i} + 1))$, conv $\lambda\mu f a \cdot \mathfrak{K}(\boldsymbol{i} + 1, \lambda\tau \cdot I^\tau(\mu(S^2(\boldsymbol{i} + 1), \mathfrak{K}(\boldsymbol{i} + 1, f, f, a))), f, a)$, conv $\lambda\mu f a \cdot \{\lambda\mu f a \cdot \mu(\boldsymbol{i} + 2, \mathfrak{K}(\boldsymbol{i}, f, f, a), \cdots, \mathfrak{K}(1, f, f, a), f(1, a), a)\}(\lambda\tau \cdot I^\tau(\mu(S^2(\boldsymbol{i} + 1), \mathfrak{K}(\boldsymbol{i} + 1, f, f, a))), f, a)$ (by

* For $\mathfrak{F}_k(1, \mathfrak{B}_{k1}), \mathfrak{F}_k(2, \mathfrak{F}_k(1, \mathfrak{B}_{k1})), \cdots$ conv $\boldsymbol{A}', \boldsymbol{F}'(1, \boldsymbol{A}'), \cdots$, respectively, where $\boldsymbol{A}' \to \mathfrak{F}_k(1, \mathfrak{B}_{k1})$ and $\boldsymbol{F}' \to \lambda x p \cdot \mathfrak{F}_k(x + 1, p)$.

the assumption), conv $\lambda\mu fa \cdot \mu(\boldsymbol{i}+3, \mathfrak{R}(\boldsymbol{i}+1, f, f, a), \mathfrak{R}(\boldsymbol{i}, f, f, a), \cdots, \mathfrak{R}(1, f, f, a), f(1, a), a)$. Hence, by induction with respect to i, $\mathfrak{R}(\boldsymbol{i}+1)$ conv $\lambda\mu fa \cdot \mu(\boldsymbol{i}+2, \mathfrak{R}(\boldsymbol{i}, f, f, a), \cdots, \mathfrak{R}(1, f, f, a), f(1, a), a)$. By 15III, there can be found an expression $\boldsymbol{L}$ which defines $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\mathfrak{R}(1, \boldsymbol{F}, \boldsymbol{F}, \boldsymbol{A}_1)$, $\mathfrak{R}(2, \boldsymbol{F}, \boldsymbol{F}, \boldsymbol{A}_1), \cdots$. Then $\boldsymbol{L}(3)$ conv $\mathfrak{B}(\boldsymbol{F}, \boldsymbol{F}, \boldsymbol{A}_1)$, conv $\boldsymbol{A}_3$. Assuming that $\boldsymbol{L}(\boldsymbol{j})$ conv $\boldsymbol{A}_j$ $(j = 1, \cdots, i+2)$, then $\boldsymbol{L}(\boldsymbol{i}+3)$ conv $\mathfrak{R}(\boldsymbol{i}+1, \boldsymbol{F}, \boldsymbol{F}, \boldsymbol{A}_1)$, conv $\{\lambda\mu fa \cdot \mu(\boldsymbol{i}+2, \mathfrak{R}(\boldsymbol{i}, f, f, a), \cdots, \mathfrak{R}(1, f, f, a), f(1, a), a)\}(\boldsymbol{F}, \boldsymbol{F}, \boldsymbol{A}_1)$ (as shown above), conv $\boldsymbol{F}(\boldsymbol{i}+2, \boldsymbol{L}(\boldsymbol{i}+2), \cdots, \boldsymbol{L}(3), \boldsymbol{L}(2), \boldsymbol{L}(1))$, conv $\boldsymbol{F}(\boldsymbol{i}+2, \boldsymbol{A}_{i+2}, \cdots, \boldsymbol{A}_1)$ (by hyp.), which is $\boldsymbol{A}_{i+3}$. Hence, by induction, $\boldsymbol{L}(\boldsymbol{i})$ conv $\boldsymbol{A}_i$ $(i = 1, 2, \cdots)$.

In 15III-15VII, the expressions $\boldsymbol{F}, \boldsymbol{A}, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_k$ of the hypotheses may be replaced by any definable functions of given numbers of positive integers. For example, 15V can be generalized thus: *If the free symbols of* $\boldsymbol{F}$ *are included among those of* $\boldsymbol{A}$, *then a formula* $\boldsymbol{L}$ *can be found such that* $\boldsymbol{L}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_m, 1)$ *conv* $\boldsymbol{A}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$ *and* $\boldsymbol{L}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_m, \boldsymbol{i}+1)$ *conv* $\boldsymbol{F}(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_m, \boldsymbol{i}, \boldsymbol{L}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_m, \boldsymbol{i}))$ $(x_1, \cdots, y_m = 1, 2, \cdots)$. For if $\boldsymbol{A}' \rightarrow I^{\boldsymbol{b}_1}(\cdots I^{\boldsymbol{b}_m}(\boldsymbol{A}(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n))\cdots)$ and $\boldsymbol{F}' \rightarrow I^{\boldsymbol{a}_1}(\cdots I^{\boldsymbol{a}_n}(\boldsymbol{F}(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m))\cdots)$, where $\boldsymbol{a}_1, \cdots, \boldsymbol{b}_m$ are distinct proper symbols, then, by 15V, there exists an expression $\boldsymbol{L}'$ which defines $\boldsymbol{A}'$, $\boldsymbol{F}'(1, \boldsymbol{A}')$, $\cdots$, and we may take for $\boldsymbol{L}$ the function $\lambda\boldsymbol{r}_1 \cdots \boldsymbol{s}_m\boldsymbol{p} \cdot \{\lambda\boldsymbol{a}_1 \cdots \boldsymbol{b}_m \cdot \boldsymbol{L}'(\boldsymbol{p})\}(\boldsymbol{r}_1, \cdots, \boldsymbol{s}_m)$. Any of the parameters $\boldsymbol{x}_1, \cdots, \boldsymbol{y}_m$ of the sequence defined by $\boldsymbol{L}(\boldsymbol{x}_1, \cdots, \boldsymbol{y}_m)$ may be equated, since *a function obtained from a definable function by equating (or interchanging) a pair of variables is definable (provided the domains of the two variables are the same)*. For if $\boldsymbol{L}$ defines $L(x, y)$, then $\lambda\boldsymbol{x} \cdot \boldsymbol{L}(\boldsymbol{x}, \boldsymbol{x})$ defines $L'(x)$ where $L'(x) = L(x, x)$ and $\lambda\boldsymbol{xy} \cdot \boldsymbol{L}(\boldsymbol{y}, \boldsymbol{x})$ defines $L''(x, y)$ where $L''(x, y) = L(y, x)$; and similarly for functions of more variables. *A function obtained from a definable function by substituting for a certain variable a definable function of other variables is definable (provided the domain of the replaced variable contains the domain of values of the substituted function).*

It is clear from the foregoing that every function recursive in the limited sense of Gödel (1931)* is definable, if we use $\lambda fx \cdot f(x)$, $S(\lambda fx \cdot f(x))$, $S(S(\lambda fx \cdot f(x))), \cdots$ as formulas for the numbers $0, 1, 2, \cdots$, resp. (thus going over from our theory of positive integers to a like theory of natural numbers), or if we replace natural numbers by positive integers in Gödel's theory. In either case Gödel's Theorems I-IV provide a convenient means

* Kurt Gödel, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I," *Monatshefte für Mathematik und Physik*, vol. 38 (1931), pp. 173-198. Cf. p. 179.

for showing that various functions, such as quotient, remainder, highest common factor, n-th prime number, are definable.*

It is also true that functions recursive in various more general senses may be defined formally.†

In some situations in which one of the above methods can be used a special device may be more expeditious.

Situations which do not come precisely within the scope of any one of the theorems of this and the following sections may often be dealt with by using several of them and by employing supplementary devices. As a general method of procedure, when it is not at once evident how to define a sequence $\boldsymbol{K}_1, \boldsymbol{K}_2, \cdots$, we attempt to find another sequence $\boldsymbol{K}'_1, \boldsymbol{K}'_2, \cdots$ and a $\boldsymbol{J}$ such that $\boldsymbol{J}(\boldsymbol{K}'_1)$ conv $\boldsymbol{K}_1$, $\boldsymbol{J}(\boldsymbol{K}'_2)$ conv $\boldsymbol{K}_2, \cdots$ and to define $\boldsymbol{K}'_1, \boldsymbol{K}'_2, \cdots$; or, more generally, to find and define two other sequences $\boldsymbol{K}'_1, \boldsymbol{K}'_2, \cdots$ and $\boldsymbol{K}''_1, \boldsymbol{K}''_2, \cdots$ such that $\boldsymbol{K}''_1(\boldsymbol{K}'_1)$ conv $\boldsymbol{K}_1$, $\boldsymbol{K}''_2(\boldsymbol{K}'_2)$ conv $\boldsymbol{K}_2, \cdots$.

In case there is given a recursive situation like that in one of our theorems but with the function relating the members of the sequence in intuitive logic, the difficulty of finding a function $\boldsymbol{F}$ of the formal logic relating the members may often be evaded by the introduction into the terms of the sequence of an extra bound symbol on which a substitution can be made which transforms any member of the sequence $\boldsymbol{K}'_1, \boldsymbol{K}'_2, \cdots$ thus obtained into the next member.

Given a positive integer n, let n_0 denote n^n, and n_{k+1} denote $(\cdots(n_k)_k\cdots)_k$ (n_k subscripts). n_n as a function of n is defined formally by $\mathfrak{Z}$ if $\mathfrak{Z} \rightarrow \lambda n \cdot [\lambda \rho m \cdot \rho^{\rho(m)}(m)]^n(\lambda r \cdot r^r, n)$. It is amazing that such a brief formula as $\mathfrak{Z}(3)$ should have so long a normal form (cf. § C5).

16. Finite sums and products. Let $\mathfrak{f} \rightarrow \lambda \pi \rho f m \cdot \rho(f, m) + f(m + \pi)$. By 15V, the sequence $1, \mathfrak{f}(1,1), \mathfrak{f}(2, \mathfrak{f}(1,1)), \cdots$ is definable by a formula $\mathfrak{S}$

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* In the first case, it should be noted at the outset that sum, product, difference, etc., are definable in the resulting theory of natural numbers.

In the second case, the absence of 0 causes no difficulty in proving Gödel's I-IV (as modified in statement by the change from natural numbers to positive integers), since o may be used to multiply 1's and 2's as 0's and 1's, respectively (cf. 15Ij).

† As an example, given formulas $\boldsymbol{F}$ and $\boldsymbol{G}$ having the same free symbols, to obtain a formula $\boldsymbol{H}$ such that $\boldsymbol{H}(1, \boldsymbol{n})$ conv $\boldsymbol{F}(\boldsymbol{n})$, $\boldsymbol{H}(\boldsymbol{m}+1, 1)$ conv $\boldsymbol{G}(\boldsymbol{m})$, and $\boldsymbol{H}(\boldsymbol{m}+1, \boldsymbol{n}+1)$ conv $\boldsymbol{H}(\boldsymbol{m}, \boldsymbol{H}(\boldsymbol{m}+1, \boldsymbol{n}))$ $(m, n = 1, 2, \cdots)$, we may use 15III-15V, according to which formulas $\boldsymbol{L}$, $\mathfrak{R}$, and $\mathfrak{H}$ can be found such that $\boldsymbol{L}(1)$ conv $\boldsymbol{F}$, $\boldsymbol{L}(2)$ conv $\boldsymbol{G}$, $\mathfrak{R}(1)$ conv $\lambda hxyl \cdot h(1, Iy, I, l(2, x))$, $\mathfrak{R}(\boldsymbol{n}+1)$ conv $\lambda hxyl \cdot h(\mathfrak{R}(\boldsymbol{n}, h, x, y-1, l), l)$, $\mathfrak{H}(1)$ conv $\lambda yl \cdot l(1, y)$, and $\mathfrak{H}(\boldsymbol{m}+1)$ conv $\lambda y \cdot \mathfrak{R}(y, \mathfrak{H}(\boldsymbol{m}), \boldsymbol{m}, y)$, and let $\boldsymbol{H} \rightarrow \lambda pq \cdot \mathfrak{H}(p, q, \boldsymbol{L})$. By induction with respect to m, $\mathfrak{H}(\boldsymbol{m}, 1, I, I)$ conv I; using this fact, $\boldsymbol{H}$ will be found to have the desired properties.

such that $N(\boldsymbol{X}) \vdash' \mathfrak{S}(S(\boldsymbol{X})) =' \mathfrak{f}(\boldsymbol{X}, \mathfrak{S}(\boldsymbol{X}))$. Then $\mathfrak{S}(\boldsymbol{i}, \lambda \boldsymbol{x} \cdot \boldsymbol{F}'\boldsymbol{x}), \boldsymbol{m})$ conv $\boldsymbol{F}(\boldsymbol{m}) + \boldsymbol{F}(\boldsymbol{m} + 1) + \cdots + \boldsymbol{F}([\boldsymbol{m} + \boldsymbol{i}] - 1)$ $(m, i = 1, 2, \cdots)$. Let $\sum_{\boldsymbol{x}=\boldsymbol{m}}^{\boldsymbol{n}} [\boldsymbol{R}]$ be an abbreviation for $\mathfrak{S}([\boldsymbol{n} + 1] - \boldsymbol{m}, \lambda \boldsymbol{x} \cdot \boldsymbol{R}, \boldsymbol{m})$, and define $\prod_{\boldsymbol{x}=\boldsymbol{m}}^{\boldsymbol{n}} [\boldsymbol{R}]$ similarly, replacing the first occurrence of $+$ in $\mathfrak{f}$ by $\times$.

16I. *If m and n are positive integers and $m \leq n$, then* $\sum_{\boldsymbol{x}=\boldsymbol{m}}^{\boldsymbol{n}} \boldsymbol{F}(\boldsymbol{x})$ *conv* $\boldsymbol{F}(\boldsymbol{m}) + \boldsymbol{F}(\boldsymbol{m} + 1) + \cdots + \boldsymbol{F}(\boldsymbol{n})$ *and* $\prod_{\boldsymbol{x}=\boldsymbol{m}}^{\boldsymbol{n}} F(x)$ *conv* $\boldsymbol{F}(\boldsymbol{m}) \times \boldsymbol{F}(\boldsymbol{m} + 1) \times \cdots \times \boldsymbol{F}(\boldsymbol{n})$.

16.1: $[N(\rho) \supset_\rho N(f(\rho))] \supset_f \cdot N(n) \supset_n \cdot N(\sum_{x=1}^{n} f(x))$.

16.2: $[N(\rho) \supset_\rho N(f(\rho))] \supset_f \cdot N(n) \supset_n \cdot \sum_{x=1}^{S(n)} f(x) = \sum_{x=1}^{n} f(x) + f(S(n))$.

16.3: $[N(\rho) \supset_\rho N(f(\rho))] \supset_f \cdot N(n) \supset_n \cdot S(\sum_{x=1}^{n} f(x)) > n$.

Proofs. Assume $N(\rho) \supset_\rho N(f(\rho))$. Then (1) $N(f(1))$, and by conversion, $N(\sum_{x=1}^{1} f(x))$. (2) Assume $N(n)$. Then $\sum_{x=1}^{S(n)} f(x)$ conv $\mathfrak{S}([S(n) + 1] - 1, \lambda x \cdot f(x), 1)$, $= \mathfrak{S}(S(n), \lambda x \cdot f(x), 1)$, $= \mathfrak{f}(n, \mathfrak{S}(n), \lambda x \cdot f(x), 1)$, conv $\mathfrak{S}(n, \lambda x \cdot f(x), 1) + f(1 + n)$, $= \mathfrak{S}([n + 1] - 1, \lambda x \cdot f(x), 1) + f(S(n))$, which is $\sum_{x=1}^{n} f(x) + f(S(n))$. (3) Assuming $N(n)$ and $N(\sum_{x=1}^{n} f(x))$, and using (2) and 5.2, $N(\sum_{x=1}^{S(n)} f(x))$. (4) From (1) and (3), by induction, $N(n) \supset_n \cdot N(\sum_{x=1}^{n} f(x))$. Hence $\vdash$ 16.1. (5) Assume $N(n)$. Using 16.1, $E(\sum_{x=1}^{S(n)} f(x))$. Hence, using (2) and § 2, $\sum_{x=1}^{S(n)} f(x) = \sum_{x=1}^{n} f(x) + f(S(n))$. Hence $\vdash$ 16.2. (6) By (1) and 12.8, $S(\sum_{x=1}^{1} f(x)) > 1$. Assume $N(n)$ and $S(\sum_{x=1}^{n} f(x)) > n$. Then $S^2(\sum_{x=1}^{S(n)} f(x)) = S^2(\sum_{x=1}^{n} f(x) + f(S(n)))$ (by (2)), $= S(\sum_{x=1}^{n} f(x)) + S(f(S(n)))$, $> S(\sum_{x=1}^{n} f(x)) + 1$ (12.8, 12.11), conv $S^2(\sum_{x=1}^{n} f(x))$, $> S(n)$ (by $S(\sum_{x=1}^{n} f(x)) > n$ and 12.11). Hence $S(\sum_{x=1}^{S(n)} f(x)) > S(n)$ (12.14). By induction, $N(n) \supset_n \cdot S(\sum_{x=1}^{n} f(x)) > n$.

16.4: $N(k)[N(\rho) \supset_\rho \cdot f(\rho) = k] \supset_{fk} \cdot N(n) \supset_n \cdot \sum_{x=1}^{n} f(x) = nk$.

Proof. Assume $N(k)\cdot N(\rho)\supset_\rho\cdot f(\rho)=k$. Then $\sum_{x=1}^{1} f(x)$ conv $f(1)$, $=k$, $=1k$ (6.1); and, assuming $N(n)$ and $\sum_{x=1}^{n} f(x)=nk$, $\sum_{x=1}^{S(n)} f(x)=\sum_{x=1}^{n} f(x)$ $+f(S(n))$ (16.2), $=nk+k$, $=S(n)k$. By induction, $N(n)\supset_n\cdot\sum_{x=1}^{n} f(x)=nk$.

17. Formal definition: successions of finite sequences. By 15III, we can find a $\mathfrak{U}$ such that $\mathfrak{U}(1)$ conv $\lambda rpqm\cdot m(\delta_{S(q)}^{r(p)}, p, S(q))$ and $\mathfrak{U}(2)$ conv $\lambda rpqm\cdot I^q(m(\delta_1^{r(S(p))}, S(p), 1))$. Then, by 15IV, we can find a $\mathfrak{B}$ such that $\mathfrak{B}(1)$ conv $\lambda rm\cdot m(\delta_1^{r(1)}, 1, 1)$, $\mathfrak{B}(\boldsymbol{k}+1)$ conv $\lambda r\cdot\mathfrak{B}(\boldsymbol{k}, r, \lambda\pi\cdot\mathfrak{U}(\pi, r))$ $(k=1, 2, \cdots)$, and $N(\mathbf{X})\vdash'\mathfrak{B}(S(\mathbf{X}))=' \lambda\boldsymbol{r}\cdot\mathfrak{B}(\mathbf{X}, \boldsymbol{r}, \lambda\pi\cdot\mathfrak{U}(\pi, \boldsymbol{r}))$. Let $\mathfrak{Q}\rightarrow\lambda frn\cdot\mathfrak{B}(n, r, \lambda uvw\cdot I^u(f(v, w)))$.

17I. *If* $\boldsymbol{R}$ *defines the sequence* $r_1, r_2, \cdots$ *of positive integers, then* $\mathfrak{Q}(\boldsymbol{F}, \boldsymbol{R})$ *defines the sequence* $\boldsymbol{F}(1,1)$, $\boldsymbol{F}(1,2), \cdots, \boldsymbol{F}(1, \boldsymbol{r}_1)$, $\boldsymbol{F}(2,1)$, $\boldsymbol{F}(2,2), \cdots,$ $\boldsymbol{F}(2, \boldsymbol{r}_2), \cdots$.

For, under the hypothesis, $\lambda n\cdot\mathfrak{B}(n, \boldsymbol{R})$ defines the sequence $\lambda m\cdot m(1,1,1)$, $\lambda m\cdot m(1,1,2), \cdots, \lambda m\cdot m(1,1,\boldsymbol{r}_1-1)$, $\lambda m\cdot m(2,1,\boldsymbol{r}_1)$, $\lambda m\cdot m(1,2,1)$, $\lambda m\cdot m(1,2,2), \cdots, \lambda m\cdot m(1,2,\boldsymbol{r}_2-1)$, $\lambda m\cdot m(2,2,\boldsymbol{r}_2), \cdots$, from which fact the conclusion follows.

17.1: $$[N(\xi)\supset_\xi N(r(\xi))]\supset_r\cdot N(p)\supset_p$$
$$\cdot[x<S(p)\supset_x\cdot y<S(r(x))\supset_y\cdot t(f(x,y))]\supset_{ft}$$
$$\cdot z<S(\sum_{i=1}^{p} r(i))\supset_z\cdot t(\mathfrak{Q}(f, r, z)).$$

Proof. Note that $N(\boldsymbol{l})N(\boldsymbol{p})N(\boldsymbol{q})\vdash E(\lambda\boldsymbol{m}\cdot\boldsymbol{m}(\boldsymbol{l}, \boldsymbol{p}, \boldsymbol{q}))$. Assume $N(\xi)\supset_\xi N(r(\xi))$.

(i) Let $\mathfrak{C}_r\rightarrow\lambda p\sigma\cdot\mathfrak{B}([\sum_{i=1}^{p} r(i)+\min(\sigma, r(p))]-r(p), r)=\lambda m$ $\cdot m(\delta_{\min(\sigma, r(p))}^{r(p)}, p, \min(\sigma, r(p)))$. (1a) $\mathfrak{B}([\sum_{i=1}^{1} r(i)+\min(1, r(1))]$ $-r(1), r)$ conv $\mathfrak{B}([r(1)+\min(1, r(1))]-r(1), r)$, $=\mathfrak{B}(1, r)$, conv $\lambda m\cdot m(\delta_1^{r(1)}, 1, 1)$, $=\lambda m\cdot m(\delta_{\min(1, r(1))}^{r(1)}, 1, \min(1, r(1)))$. Thus $\mathfrak{C}_r(1,1)$. (b) Assume $N(\sigma)$ and $\mathfrak{C}_r(1, \sigma)$. Case 1: $\epsilon_{r(1)}^{S(\sigma)}=2$. Then $S(\sigma)>r(1)$; consequently $\min(\sigma, r(1))=r(1)$, $=\min(S(\sigma), r(1))$; and hence $\mathfrak{C}_r(1, S(\sigma))$ follows from $\mathfrak{C}_r(1, \sigma)$. Case 2: $\epsilon_{r(1)}^{S(\sigma)}=1$. Then $S(r(1))>S(\sigma)$, $r(1)>\sigma$, $\min(S(\sigma), r(1))=S(\sigma)$, and $\min(\sigma, r(1))=\sigma$. Hence $\mathfrak{B}([\sum_{i=1}^{1} r(i)+\min(S(\sigma), r(1))]-r(1), r)=\mathfrak{B}(\min(S(\sigma), r(1)), r)$,

$= \mathfrak{B}(S(\sigma), r)$, $= \mathfrak{B}(S(\min(\sigma, r(1))), r)$, $= \mathfrak{B}(\min(\sigma, r(1)), r, \lambda\pi \cdot \mathfrak{U}(\pi, r))$ (using the definition of $\mathfrak{B}$), $= \mathfrak{B}([\sum_{i=1}^{1} r(i) + \min(\sigma, r(1))] - r(1), r, \lambda\pi \cdot \mathfrak{U}(\pi, r))$, $= \{\lambda m \cdot m(\delta^{r(1)}_{\min(\sigma, r(1))}, 1, \min(\sigma, r(1)))\}(\lambda\pi \cdot \mathfrak{U}(\pi, r))$ (by $\mathfrak{C}_r(1, \sigma)$), $= \{\lambda m \cdot m(\delta_\sigma{}^{r(1)}, 1, \sigma)\}(\lambda\pi \cdot \mathfrak{U}(\pi, r))$, conv $\mathfrak{U}(\delta_\sigma{}^{r(1)}, r, 1, \sigma)$, $= \mathfrak{U}(1, r, 1, \sigma)$ (since $\sigma < r(1)$), conv $\lambda m \cdot m(\delta^{r(1)}_{S(\sigma)}, 1, S(\sigma))$ (using the def. of $\mathfrak{U}$), $= \lambda m \cdot m(\delta^{r(1)}_{\min(S(\sigma), r(1))}, 1, \min(S(\sigma), r(1)))$. Thus $\mathfrak{C}_r(1, S(\sigma))$. Hence, by cases (C9I), $\mathfrak{C}_r(1, S(\sigma))$. (c) From (a) and (b) by induction, $N(\sigma) \supset_\sigma \cdot \mathfrak{C}_r(1, \sigma)$. (2) Assume $N(p)$ and $N(\sigma) \supset_\sigma \cdot \mathfrak{C}_r(p, \sigma)$. (a) $\mathfrak{B}([\sum_{i=1}^{S(p)} r(i) + \min(1, r(S(p)))] - r(S(p)), r) = \mathfrak{B}(S(\sum_{i=1}^{p} r(i)), r)$ (16.2, 11.2, 5.4, 13.2, 12.8), $= \mathfrak{B}(\sum_{i=1}^{p} r(i), r, \lambda\pi \cdot \mathfrak{U}(\pi, r))$ (by the def. of $\mathfrak{B}$), $= \mathfrak{B}([\sum_{i=1}^{p} r(i) + r(p)] - r(p), r, \lambda\pi \cdot \mathfrak{U}(\pi, r))$, $= \mathfrak{B}([\sum_{i=1}^{p} r(i) + \min(r(p), r(p))] - r(p), r, \lambda\pi \cdot \mathfrak{U}(\pi, r))$, $= \{\lambda m \cdot m(\delta^{r(p)}_{\min(r(p), r(p))}, p, \min(r(p), r(p)))\}(\lambda\pi \cdot \mathfrak{U}(\pi, r))$ (by $N(\sigma) \supset_\sigma \mathfrak{C}_r(p, \sigma)$ and $N(r(p))$), $= \{\lambda m \cdot m(2, p, r(p))\}(\lambda\pi \cdot \mathfrak{U}(\pi, r))$, conv $\mathfrak{U}(2, r, p, r(p))$, conv $\lambda m \cdot I^{r(p)}(m(\delta_1{}^{r(S(p))}, S(p), 1))$ (using the def. of $\mathfrak{U}$), $= \lambda m \cdot m(\delta_1{}^{r(S(p))}, S(p), 1)$, $= \lambda m \cdot m(\delta^{r(S(p))}_{\min(1, r(S(p)))}, S(p), \min(1, r(S(p))))$. Thus $\mathfrak{C}_r(S(p), 1)$. (b) Assuming $N(\sigma)$ and $\mathfrak{C}_r(S(p), \sigma)$, $\mathfrak{C}_r(S(p), S(\sigma))$ follows by reasoning like that used in (1b) (in Case 2, 16.2 is used). (c) From (a) and (b) by induction, $N(\sigma) \supset_\sigma \mathfrak{C}_r(S(p), \sigma)$. (3) From (1) and (2) by induction, $N(p) \supset_p \cdot N(\sigma) \supset_\sigma \mathfrak{C}_r(p, \sigma)$. Thence we can infer $N(p) \supset_p \cdot \sigma < S(r(p)) \supset_\sigma \cdot \mathfrak{B}([\sum_{i=1}^{p} r(i) + \sigma] - r(p), r) = \lambda m \cdot m(\delta_\sigma{}^{r(p)}, p, \sigma)$.

(ii) Let $\mathfrak{L}_{rz} \rightarrow \lambda p \cdot \sum ab \cdot a < S(p) \cdot b < S(r(a)) \cdot z = [\sum_{i=1}^{a} r(i) + b] - r(a)$. (a) Assume $z < S(\sum_{i=1}^{1} r(i))$. Then $z = [r(1) + z] - r(1)$, conv $[\sum_{i=1}^{1} r(i) + z] - r(1)$; also $1 < S(1)$ and $z < S(\sum_{i=1}^{1} r(i))$, conv $S(r(1))$. Hence, using Axiom 14 and Rule IV, $\mathfrak{L}_{rz}(1)$. By Theorem I, $z < S(\sum_{i=1}^{1} r(i)) \supset_z \mathfrak{L}_{rz}(1)$. (b) Assume $N(p)$, $z < S(\sum_{i=1}^{p} r(i)) \supset_z \mathfrak{L}_{rz}(p)$, and $z < S(\sum_{i=1}^{S(p)} r(i))$. Case 1: $\epsilon(S(\sum_{i=1}^{p} r(i)), z) = 2$. Then $z < S(\sum_{i=1}^{p} r(i))$, and, using $z < S(\sum_{i=1}^{p} r(i)) \supset_z \mathfrak{L}_{rz}(p)$, we can prove $\mathfrak{L}_{rz}(S(p))$ by means of the second clause of Theorem I. Case 2: $\epsilon(S(\sum_{i=1}^{p} r(i)), z) = 1$. Then $z > \sum_{i=1}^{p} r(i)$, and

hence $z = \sum_{i=1}^{p} r(i) + \cdot z - \sum_{i=1}^{p} r(i)$ (12.5), $= [\sum_{i=1}^{S(p)} r(i) + \cdot z - \sum_{i=1}^{p} r(i)] - r(S(p))$ (16.2, 11.2, 5.4). Case A: $\epsilon(z - \sum_{i=1}^{p} r(i), r(S(p))) = 1$. Then $z - \sum_{i=1}^{p} r(i) < S(r(S(p)))$. Case B: $\epsilon(z - \sum_{i=1}^{p} r(i), r(S(p))) = 2$. Then $z - \sum_{i=1}^{p} r(i) > r(S(p))$. Hence $\sum_{i=1}^{S(p)} r(i) = \sum_{i=1}^{p} r(i) + r(S(p))$ (16.2), $< \sum_{i=1}^{p} r(i) + \cdot z - \sum_{i=1}^{p} r(i)$ (12.11), $= z$. Hence $\epsilon(z, \sum_{i=1}^{S(p)} r(i)) = 2$. But $\epsilon(z, \sum_{i=1}^{S(p)} r(i)) = 1$ is a consequence of the assumption $z < S(\sum_{i=1}^{S(p)} r(i))$. Hence, by cases A and B and *reductio ad absurdum* (C10II), $z - \sum_{i=1}^{p} r(i) < S(r(S(p)))$. Also $S(p) < S^2(p)$. Hence, using Axiom 14 and Rule IV, $\mathfrak{L}_{rz}(S(p))$. Hence, by cases 1 and 2 (C9I), $\mathfrak{L}_{rz}(S(p))$. By Thm. I, $z < S(\sum_{i=1}^{S(p)} r(i)) \supset_z \mathfrak{L}_{rz}(S(p))$.
(c) From (a) and (b) by induction, $N(p) \supset_p \cdot z < S(\sum_{i=1}^{p} r(i)) \supset_z \mathfrak{L}_{rz}(p)$.

(iii) Assume $N(p)$, $x < S(p) \supset_x \cdot y < S(r(x)) \supset_y \cdot t(f(x,y))$, and $z < S(\sum_{i=1}^{p} r(i))$. Then, by (ii), $\sum ab \cdot a < S(p) \cdot b < S(r(a)) \cdot z = [\sum_{i=1}^{a} r(i) + b] - r(a)$. Assume $a < S(p) \cdot b < S(r(a)) \cdot z = [\sum_{i=1}^{a} r(i) + b] - r(a)$. Then $\mathfrak{Q}(f, r, z)$ conv $\mathfrak{B}(z, r, \lambda uvw \cdot I^u(f(v,w)))$, $= \mathfrak{B}([\sum_{i=1}^{a} r(i) + b] - r(a), r, \lambda uvw \cdot I^u(f(v,w)))$, $= \{\lambda m \cdot m(\delta_b{}^{r(a)}, a, b)\}(\lambda uvw \cdot I^u(f(v,w)))$ (by (i)), conv $\delta_b{}^{r(a)}(I, f(a,b))$, $= f(a,b)$ (7.2). Moreover $t(f(a,b))$ is provable from our assumptions. Hence $t(\mathfrak{Q}(f,r,z))$. By the second clause of Theorem I, $t(\mathfrak{Q}(f,r,z))$ is provable without the last assumption.

17.2: $[N(\xi) \supset_\xi N(r(\xi))] \supset_r \cdot [N(x) \supset_x \cdot y < S(r(x)) \supset_y \cdot t(f(x,y))] \supset_{ft} \cdot N(z) \supset_z \cdot t(\mathfrak{Q}(f,r,z))$.

Proof. Assuming $N(\xi) \supset_\xi N(r(\xi))$, $N(x) \supset_x \cdot y < S(r(x)) \supset_y \cdot t(f(x,y))$, and $N(z)$, we can prove $x < S(z) \supset_x \cdot y < S(r(x)) \supset_y \cdot t(f(x,y))$, and also, using 16.3, $z < S(\sum_{i=1}^{z} r(i))$. Hence, by 17.1, $t(\mathfrak{Q}(f,r,z))$.

Using 17I, the dyads (triads, $\cdots$) of positive integers can be *enumerated formally* (*i. e.*, there is an enumeration of them which is definable formally). As another application of $\mathfrak{Q}$, we establish the following theorem:

17II. *If* $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_l, \boldsymbol{R}_1, \cdots, \boldsymbol{R}_{m+n}$ *contain no free symbols, then a*

formula $\boldsymbol{H}$ *can be found such that* (1) $\boldsymbol{H}$ *enumerates formally* (*with repetitions*) *the formulas derivable from* $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_l$ *by zero or more operations of passing from* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *to* $\boldsymbol{R}_1(\boldsymbol{A}), \cdots, \boldsymbol{R}_m(\boldsymbol{A}), \boldsymbol{R}_{m+1}(\boldsymbol{A}, \boldsymbol{B}), \cdots$, *or* $\boldsymbol{R}_{m+n}(\boldsymbol{A}, \boldsymbol{B})$, *and* (2) $\boldsymbol{T}(\boldsymbol{A}_1), \cdots, \boldsymbol{T}(\boldsymbol{A}_l), \boldsymbol{T}(\boldsymbol{a}) \supset_{\boldsymbol{a}} \boldsymbol{T}(\boldsymbol{R}_1(\boldsymbol{a})), \cdots, \boldsymbol{T}(\boldsymbol{a}) \supset_{\boldsymbol{a}} \boldsymbol{T}(\boldsymbol{R}_m(\boldsymbol{a})), \boldsymbol{T}(\boldsymbol{a})\boldsymbol{T}(\boldsymbol{b}) \supset_{\boldsymbol{ab}} \boldsymbol{T}(\boldsymbol{R}_{m+1}(\boldsymbol{a}, \boldsymbol{b})), \cdots, \boldsymbol{T}(\boldsymbol{a})\boldsymbol{T}(\boldsymbol{b}) \supset_{\boldsymbol{ab}} \boldsymbol{T}(\boldsymbol{R}_{m+n}(\boldsymbol{a}, \boldsymbol{b})) \vdash \boldsymbol{N}(\boldsymbol{z}) \supset_{\boldsymbol{z}} \boldsymbol{T}(\boldsymbol{H}(\boldsymbol{z}))$.

Proof. Let $\boldsymbol{A}_{1i} \rightarrow \boldsymbol{A}_i$ $(i = 1, \cdots, l_1$, where $l_1 = l)$. Given $\boldsymbol{A}_{ki}$ $(i = 1, \cdots, l_k)$, let $\boldsymbol{A}_{k+1,1}, \cdots, \boldsymbol{A}_{k+1, l_{k+1}}$, where $l_{k+1} = (1 + m + n) l_k{}^2$, be the formulas $\boldsymbol{A}_{k1}, \cdots, \boldsymbol{A}_{kl_k}, \cdots, \boldsymbol{A}_{k1}, \cdots, \boldsymbol{A}_{kl_k}$; $\boldsymbol{R}_1(\boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_1(\boldsymbol{A}_{kl_k}), \cdots, \boldsymbol{R}_1(\boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_1(\boldsymbol{A}_{kl_k})$; $\cdots$; $\boldsymbol{R}_m(\boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_m(\boldsymbol{A}_{kl_k}), \cdots, \boldsymbol{R}_m(\boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_m(\boldsymbol{A}_{kl_k})$; $\boldsymbol{R}_{m+1}(\boldsymbol{A}_{k1}, \boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_{m+1}(\boldsymbol{A}_{k1}, \boldsymbol{A}_{kl_k}), \cdots, \boldsymbol{R}_{m+1}(\boldsymbol{A}_{kl_k}, \boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_{m+1}(\boldsymbol{A}_{kl_k}, \boldsymbol{A}_{kl_k})$; $\cdots$; $\boldsymbol{R}_{m+n}(\boldsymbol{A}_{k1}, \boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_{m+n}(\boldsymbol{A}_{k1}, \boldsymbol{A}_{kl_k}), \cdots, \boldsymbol{R}_{m+n}(\boldsymbol{A}_{kl_k}, \boldsymbol{A}_{k1}), \cdots, \boldsymbol{R}_{m+n}(\boldsymbol{A}_{kl_k}, \boldsymbol{A}_{kl_k})$ $(1 + m + n$ sets of l_k sets of l_k formulas each), respectively. Then the sequence of formulas $\boldsymbol{A}_{k1}, \cdots, \boldsymbol{A}_{kl_k}$ (defined by induction with respect to k) is an enumeration (with repetitions) of the formulas derivable from $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_l$ by not more than $k - 1$ applications of the operations under consideration.

By 15III, there can be found a formula $\boldsymbol{F}_1$ such that $\boldsymbol{F}_1(\boldsymbol{i})$ conv $\boldsymbol{A}_{1i}$ $(i = 1, \cdots, l_1)$, and a formula $\boldsymbol{J}$ which defines the finite sequence $\lambda fji \cdot I^j(f(i))$, $\lambda fji \cdot \boldsymbol{R}_1(I^j(f(i))), \cdots, \lambda fji \cdot \boldsymbol{R}_m(I^j(f(i))), \lambda fji \cdot \boldsymbol{R}_{m+1}(f(j), f(i)), \cdots$, $\lambda fji \cdot \boldsymbol{R}_{m+n}(f(j), f(i))$. By 15IV, the sequence $l_1, l_2, l_3, \cdots$ can be defined by a formula $\boldsymbol{L}$ such that $N(\boldsymbol{X}) \vdash' \boldsymbol{L}(S(\boldsymbol{X})) =' [1 + \boldsymbol{m} + \boldsymbol{n}]\boldsymbol{L}(\boldsymbol{X})\boldsymbol{L}(\boldsymbol{X})$. If $\boldsymbol{F}_{k+1} \rightarrow \mathfrak{Q}\ (\lambda v \cdot \mathfrak{Q}\ (\boldsymbol{J}(v, \boldsymbol{F}_k), \lambda w \cdot I^w(\boldsymbol{L}(\boldsymbol{k}))), \lambda w \cdot I^w(\boldsymbol{L}(\boldsymbol{k})\boldsymbol{L}(\boldsymbol{k})))$ $(k = 1, 2, \cdots)$, then, by 15V, the sequence $\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots$ is definable by a formula $\boldsymbol{F}$ such that $N(\boldsymbol{Y}) \vdash' \boldsymbol{F}(S(\boldsymbol{Y})) =' \mathfrak{Q}\ (\lambda \boldsymbol{v} \cdot \mathfrak{Q}\ (\boldsymbol{J}(\boldsymbol{v}, \boldsymbol{F}(\boldsymbol{Y})), \lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{Y}))), \lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{Y})\boldsymbol{L}(\boldsymbol{Y})))$. Let $\boldsymbol{H} \rightarrow \mathfrak{Q}(\boldsymbol{F}, \boldsymbol{L})$.

Assuming that $\boldsymbol{F}_k(\boldsymbol{i})$ conv $\boldsymbol{A}_{ki}$ $(i = 1, \cdots, l_k)$, it follows by 17I and the definitions of $\boldsymbol{F}_{k+1}$, $\boldsymbol{J}$ and $\boldsymbol{L}$ that $\boldsymbol{F}_{k+1}(\boldsymbol{i})$ conv $\boldsymbol{A}_{k+1,i}$ $(i = 1, \cdots, l_{k+1})$. By induction with respect to k, $\boldsymbol{F}_k(\boldsymbol{i})$ conv $\boldsymbol{A}_{ki}$ $(i = 1, \cdots, l_k;\ k = 1, 2, \cdots)$. Hence, by 17I and the definitions of $\boldsymbol{H}$, $\boldsymbol{F}$ and $\boldsymbol{L}$, $\boldsymbol{H}$ defines $\boldsymbol{A}_{11}, \cdots, \boldsymbol{A}_{1l_1}$, $\boldsymbol{A}_{21}, \cdots, \boldsymbol{A}_{2l_2}, \cdots$. Hence (1) is satisfied.

Assume $\boldsymbol{T}(\boldsymbol{A}_1), \cdots, \boldsymbol{T}(\boldsymbol{A}_l), \boldsymbol{T}(\boldsymbol{a}) \supset_{\boldsymbol{a}} \boldsymbol{T}(\boldsymbol{R}_1(\boldsymbol{a})), \cdots, \boldsymbol{T}(\boldsymbol{a}) \supset_{\boldsymbol{a}} \boldsymbol{T}(\boldsymbol{R}_m(\boldsymbol{a}))$, $\boldsymbol{T}(\boldsymbol{a})\boldsymbol{T}(\boldsymbol{b}) \supset_{\boldsymbol{ab}} \boldsymbol{T}(\boldsymbol{R}_{m+1}(\boldsymbol{a}, \boldsymbol{b})), \cdots, \boldsymbol{T}(\boldsymbol{a})\boldsymbol{T}(\boldsymbol{b}) \supset_{\boldsymbol{ab}} \boldsymbol{T}(\boldsymbol{R}_{m+n}(\boldsymbol{a}, \boldsymbol{b}))$. In the following we suppose $\boldsymbol{q}$, $\boldsymbol{x}$ and $\boldsymbol{y}$ to represent variables distinct from each other and from the variables of $\boldsymbol{T}$. (1) $N(\xi) \supset_{\xi} N(\boldsymbol{L}(\xi))$ can be proved by induction. (2) Using $\boldsymbol{T}(\boldsymbol{A}_1), \cdots, \boldsymbol{T}(\boldsymbol{A}_l)$, we can prove $N(\boldsymbol{y}) \supset_{\boldsymbol{y}} \boldsymbol{T}(\boldsymbol{F}_1(\min(\boldsymbol{y}, \boldsymbol{l})))$ by induction from an l-tuple basis, and thence infer $\boldsymbol{y} < S(\boldsymbol{l}) \supset_{\boldsymbol{y}} \boldsymbol{T}(\boldsymbol{F}_1(\boldsymbol{y}))$ by use of Theorem I and 13.2. By conversion, $\boldsymbol{y} < S(\boldsymbol{L}(1)) \supset_{\boldsymbol{y}} \boldsymbol{T}(\boldsymbol{F}(1, \boldsymbol{y}))$.

(3) Assume $N(\boldsymbol{q})$ and $\boldsymbol{y} < S(\boldsymbol{L}(\boldsymbol{q})) \supset_{\boldsymbol{y}} \boldsymbol{T}(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y}))$. (a) Assuming $\boldsymbol{x} < S(\boldsymbol{L}(\boldsymbol{q}))$ and $\boldsymbol{y} < S(\boldsymbol{L}(\boldsymbol{q}))$, we can infer $\boldsymbol{T}(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{x}))$ and $\boldsymbol{T}(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y}))$; thence, using $\boldsymbol{T}(\boldsymbol{a}) \supset_{\boldsymbol{a}} \boldsymbol{T}(\boldsymbol{R}_c(\boldsymbol{a}))$, $\boldsymbol{T}(\boldsymbol{R}_c(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y})))$ $(c = 1, \cdots, m)$, and, using $\boldsymbol{T}(\boldsymbol{a})\boldsymbol{T}(\boldsymbol{b}) \supset_{\boldsymbol{ab}} \boldsymbol{T}(\boldsymbol{R}_{m+d}(\boldsymbol{a}, \boldsymbol{b}))$, $\boldsymbol{T}(\boldsymbol{R}_{m+d}(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{x}), \boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y})))$ $(d = 1, \cdots, n)$; also, using the definition of $\boldsymbol{J}$, $\boldsymbol{J}(1, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y})$, $\boldsymbol{J}(1 + \boldsymbol{c}, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{R}_c(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y}))$, and $\boldsymbol{J}(1 + \boldsymbol{m} + \boldsymbol{d}, \boldsymbol{F}(\boldsymbol{q})\ \boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{R}_{m+d}(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{x}), \boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y}))$; hence $\boldsymbol{T}(\boldsymbol{J}(1, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}))$, $\boldsymbol{T}(\boldsymbol{J}(1 + \boldsymbol{c}, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}))$, $\boldsymbol{T}(\boldsymbol{J}(1 + \boldsymbol{m} + \boldsymbol{d}, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}))$. Thus, for $j = 1, \cdots, 1 + m + n$, $\boldsymbol{T}(\boldsymbol{J}(\boldsymbol{j}, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}))$ is a consequence of $\boldsymbol{x} < S(\boldsymbol{L}(\boldsymbol{q}))$, $\boldsymbol{y} < S(\boldsymbol{L}(\boldsymbol{q}))$ and our other assumptions. Using these relations, we can prove by means of Thm. I and induction from a $1 + m + n$-tuple basis, $N(\boldsymbol{v}) \supset_{\boldsymbol{v}} \cdot \boldsymbol{x} < S(\boldsymbol{L}(\boldsymbol{q})) \supset_{\boldsymbol{x}} \cdot \boldsymbol{y} < S(\boldsymbol{L}(\boldsymbol{q})) \supset_{\boldsymbol{y}} \cdot \boldsymbol{T}(\boldsymbol{J}(\min(\boldsymbol{v}, 1 + \boldsymbol{m} + \boldsymbol{n}), \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}))$. (b) Assume $\boldsymbol{v} < S(1 + \boldsymbol{m} + \boldsymbol{n})$. Then from (a), by means of Theorem I, 13.2 and 7.2, we obtain $\boldsymbol{x} < S(\boldsymbol{L}(\boldsymbol{q})) \supset_{\boldsymbol{x}} \cdot \boldsymbol{y} < S(\{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{x})) \supset_{\boldsymbol{y}} \cdot \boldsymbol{T}(\boldsymbol{J}(\boldsymbol{v}, \boldsymbol{F}(\boldsymbol{q}), \boldsymbol{x}, \boldsymbol{y}))$. Using (1), $N(\boldsymbol{L}(\boldsymbol{q}))$; and hence, using 7.2 and Theorem I, $N(\boldsymbol{s}) \supset_{\boldsymbol{s}} N(\{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{s}))$. These results with 17.1 yield $\boldsymbol{z} < S(\sum_{\boldsymbol{u}=1}^{\boldsymbol{L}(\boldsymbol{q})} \{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{u})) \supset_{\boldsymbol{z}} \boldsymbol{T}(\mathfrak{Q}(\boldsymbol{J}(\boldsymbol{v}, \boldsymbol{F}(\boldsymbol{q})), \lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})), \boldsymbol{z}))$. Also, by using $N(\boldsymbol{L}(\boldsymbol{q}))$, 7.2 and Theorem I, $N(\boldsymbol{L}(\boldsymbol{q})) \cdot N(\boldsymbol{s}) \supset_{\boldsymbol{s}} \{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{s}) = \boldsymbol{L}(\boldsymbol{q})$; hence $\sum_{\boldsymbol{u}=1}^{\boldsymbol{L}(\boldsymbol{q})} \{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{u}) = \boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q})$ (by 16.4), $= \{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q})\}(\boldsymbol{v})$. Using this result with the preceding, and applying Theorem I, $\boldsymbol{v} < S(1 + \boldsymbol{m} + \boldsymbol{n}) \supset_{\boldsymbol{v}} \cdot \boldsymbol{z} < S(\{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{v})) \supset_{\boldsymbol{z}} \cdot \boldsymbol{T}(\{\lambda \boldsymbol{v} \cdot \mathfrak{Q}(\boldsymbol{J}(\boldsymbol{v}, \boldsymbol{F}(\boldsymbol{q})), \lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})))\}(\boldsymbol{v}, \boldsymbol{z}))$. (c) By Theorem I, $N(\boldsymbol{s}) \supset_{\boldsymbol{s}} N(\{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{s}))$. Using the latter, $N(1 + \boldsymbol{m} + \boldsymbol{n})$, and the result of (b) with 17.1, $\boldsymbol{z} < S(\sum_{\boldsymbol{u}=1}^{1+\boldsymbol{m}+\boldsymbol{n}} \{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{u})) \supset_{\boldsymbol{z}} \boldsymbol{T}(\mathfrak{Q}(\lambda \boldsymbol{v} \cdot \mathfrak{Q}(\boldsymbol{J}(\boldsymbol{v}, \boldsymbol{F}(\boldsymbol{q})), \lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q}))), \lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q})), \boldsymbol{z}))$. Thence, using the definition of $\boldsymbol{F}$, Rule I, and the relation $\sum_{\boldsymbol{u}=1}^{1+\boldsymbol{m}+\boldsymbol{n}} \{\lambda \boldsymbol{w} \cdot I^{\boldsymbol{w}}(\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q}))\}(\boldsymbol{u}) = [1 + \boldsymbol{m} + \boldsymbol{n}]\boldsymbol{L}(\boldsymbol{q})\boldsymbol{L}(\boldsymbol{q})$ (by 16.4), $= \boldsymbol{L}(S(\boldsymbol{q}))$ (by the def. of $\boldsymbol{L}$), we infer $\boldsymbol{y} < S(\boldsymbol{L}(S(\boldsymbol{q}))) \supset_{\boldsymbol{y}} \boldsymbol{T}(\boldsymbol{F}(S(\boldsymbol{q}), \boldsymbol{y}))$. (4) From (2) and (3) by induction, $N(\boldsymbol{q}) \supset_{\boldsymbol{q}} \cdot \boldsymbol{y} < S(\boldsymbol{L}(\boldsymbol{q})) \supset_{\boldsymbol{y}} \boldsymbol{T}(\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{y}))$. This and (1) with 17.2 yield $N(\boldsymbol{z}) \supset_{\boldsymbol{z}} \boldsymbol{T}(\mathfrak{Q}(\boldsymbol{F}, \boldsymbol{L}, \boldsymbol{z}))$, or, by the definition of $\boldsymbol{H}$, $N(\boldsymbol{z}) \supset_{\boldsymbol{z}} \boldsymbol{T}(\boldsymbol{H}(\boldsymbol{z}))$.

18. The sequence of positive integers satisfying a given condition. By 15III, there can be found a formula $\mathfrak{J}$ such that

$$(1) \quad \begin{aligned} &\mathfrak{J}(1) \text{ conv } \lambda cdk \cdot c(1, d(k+1), c, d, k+1), \\ &\mathfrak{J}(2) \text{ conv } \lambda cdk \cdot c(2, I^{d(k)}, k), \end{aligned}$$

and then a formula $\mathfrak{E}$ such that

$$\mathfrak{E}(1) \text{ conv } \mathfrak{J}, \qquad \mathfrak{E}(2) \text{ conv } I. \tag{2}$$

Let $\mathfrak{p} \rightarrow \lambda dk \cdot \mathfrak{J}(d(k), \mathfrak{E}, d, k)$.

18I. *Given a positive integer k: If $\boldsymbol{D}(\boldsymbol{k})$ conv 2, $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ conv $\boldsymbol{k}$. If $\boldsymbol{D}(\boldsymbol{k})$ conv 1, $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ conv $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k}+1)$.* Hence, *if $\boldsymbol{D}(\boldsymbol{k})$ conv $\boldsymbol{D}(\boldsymbol{k}+1)$ conv $\cdots$ conv $\boldsymbol{D}(\boldsymbol{l}-1)$ conv 1 and $\boldsymbol{D}(\boldsymbol{l})$ conv 2 $(l \geq k)$, then $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ conv $\boldsymbol{l}$.*

For if $\boldsymbol{D}(\boldsymbol{k})$ conv 2, then $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ conv $\mathfrak{J}(\boldsymbol{D}(\boldsymbol{k}), \mathfrak{E}, \boldsymbol{D}, \boldsymbol{k})$ conv $\mathfrak{J}(2, \mathfrak{E}, \boldsymbol{D}, \boldsymbol{k})$ conv $\mathfrak{E}(2, I^{\boldsymbol{D}(\boldsymbol{k})}, \boldsymbol{k})$ conv $I(I^2, \boldsymbol{k})$ conv $\boldsymbol{k}$; and if $\boldsymbol{D}(\boldsymbol{k})$ conv 1, then $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ conv $\mathfrak{J}(1, \mathfrak{E}, \boldsymbol{D}, \boldsymbol{k})$ conv $\mathfrak{E}(1, \boldsymbol{D}(\boldsymbol{k}+1), \mathfrak{E}, \boldsymbol{D}, \boldsymbol{k}+1)$ conv $\mathfrak{J}(\boldsymbol{D}(\boldsymbol{k}+1), \mathfrak{E}, \boldsymbol{D}, \boldsymbol{k}+1)$ conv $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k}+1)$.

18II. *If $\boldsymbol{D}(\boldsymbol{i})$ conv 1 for every positive integer $i \geq$ the positive integer k, then $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ has no normal form.**

Proof. A derivation of $\boldsymbol{B}$ from $\boldsymbol{A}$ by applications of I and II, including at least one of the latter, will be called a *reduction.* A conversion in which III is not used may be indicated by an accent. It will be shown in a forthcoming paper by A. Church and J. B. Rosser,† that if an expression $\boldsymbol{A}$ has a normal form, then every sequence $\boldsymbol{A}$ red $\boldsymbol{A}'$ red $\boldsymbol{A}''$ red $\cdots$ of reductions is finite; ‡ and that if $\boldsymbol{P}$ conv $\boldsymbol{Q}$, then there exists a conversion of $\boldsymbol{P}$ into $\boldsymbol{Q}$ in which all applications of III follow all applications of II.§ Hence if $\bar{\boldsymbol{A}}$ is a normal form of $\boldsymbol{A}$, $\boldsymbol{A}$ conv′ $\bar{\boldsymbol{A}}$. $\lambda cdk \cdot c(1, d(k+1), c, d, k+1)$ is a normal form of $\mathfrak{J}(1)$. Consequently $\mathfrak{J}$ has a normal form $\bar{\mathfrak{J}}$, for otherwise there would exist an infinite sequence $\mathfrak{J}$ red $\mathfrak{J}'$ red $\mathfrak{J}''$ red $\cdots$, and hence an infinite sequence $\mathfrak{J}(1)$ red $\mathfrak{J}'(1)$ red $\mathfrak{J}''(1)$ red $\cdots$. (1) and (2) hold with $\mathfrak{J}$ replaced by $\bar{\mathfrak{J}}$, and *conv* by *conv′*. Moreover $\bar{\boldsymbol{i}}+1$ conv′ $\overline{\boldsymbol{i}+1}$, and from $\boldsymbol{D}(\boldsymbol{i})$ *conv* 1 follows $\boldsymbol{D}(\bar{\boldsymbol{i}})$ *conv′* 1. Then under the hypothesis, $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{i})$ red $\mathfrak{J}(\boldsymbol{D}(\boldsymbol{i}), \mathfrak{E}, \boldsymbol{D}, \boldsymbol{i})$ conv′ $\bar{\mathfrak{J}}(\boldsymbol{D}(\bar{\boldsymbol{i}}), \mathfrak{E}, \boldsymbol{D}, \bar{\boldsymbol{i}})$ conv′ $\bar{\mathfrak{J}}(1, \mathfrak{E}, \boldsymbol{D}, \bar{\boldsymbol{i}})$ red $\mathfrak{E}(1, \boldsymbol{D}(\bar{\boldsymbol{i}}+1), \mathfrak{E}, \boldsymbol{D}, \bar{\boldsymbol{i}}+1)$ conv′ $\bar{\mathfrak{J}}(\boldsymbol{D}(\bar{\boldsymbol{i}}+1), \mathfrak{E}, \boldsymbol{D}, \bar{\boldsymbol{i}}+1)$ conv′ $\bar{\mathfrak{J}}(\boldsymbol{D}(\overline{\boldsymbol{i}+1}), \mathfrak{E}, \boldsymbol{D}, \overline{\boldsymbol{i}+1})$. Hence $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ red $\bar{\mathfrak{J}}(\boldsymbol{D}(\bar{\boldsymbol{k}}), \mathfrak{E}, \boldsymbol{D}, \bar{\boldsymbol{k}})$ red $\bar{\mathfrak{J}}(\boldsymbol{D}(\overline{\boldsymbol{k}+1}), \mathfrak{E}, \boldsymbol{D}, \overline{\boldsymbol{k}+1})$ red $\bar{\mathfrak{J}}(\boldsymbol{D}(\overline{\boldsymbol{k}+2}), \mathfrak{E}, \boldsymbol{D}, \overline{\boldsymbol{k}+2})$ red $\cdots$ *ad infinitum*, which could not be if $\mathfrak{p}(\boldsymbol{D}, \boldsymbol{k})$ had a normal form.

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* *Normal form* is defined in § C5.

† A. Church and J. B. Rosser, "Some properties of conversion."

‡ In other words, given any well-formed expression $\boldsymbol{P}$, either all or none of the sequences $\boldsymbol{P}$ red $\boldsymbol{P}'$ red $\boldsymbol{P}''$ red $\cdots$ can be continued *ad infinitum.*

§ Consequently, if $\boldsymbol{A}$ has a normal form, all normal forms of $\boldsymbol{A}$ are derivable from a given one by applications of I.

By 15IV, a formula $\mathfrak{A}$ such that $\mathfrak{A}(1)$ conv $\lambda d \cdot \mathfrak{p}(d, 1)$ and $\mathfrak{A}(\boldsymbol{n}+1)$ conv $\lambda d \cdot \mathfrak{p}(d, \mathfrak{A}(\boldsymbol{n}, d)+1)$ $(n = 1, 2, \cdots)$ can be found. Let $\mathfrak{P} \rightarrow \lambda dn \cdot \mathfrak{A}(n, d)$.

18III. *If $\boldsymbol{D}$ defines the infinite sequence $d_1, d_2, d_3, \cdots$ of* 1's *and* 2's, *and $d_{n_1}, d_{n_2}, d_{n_3}, \cdots$ is the subsequence which are* 2's, *then $\mathfrak{P}(\boldsymbol{D})$ defines the sequence $n_1, n_2, n_3, \cdots$. If the latter is a finite sequence $n_1, \cdots, n_k$ $(k \geq 0)$, then, for $i > k$, $\mathfrak{P}(\boldsymbol{D}, \boldsymbol{i})$ has no normal form.*

This result, together with 15Ij, l and above results concerning the formal definability and enumerability of n-tuples of positive integers, leads to the following:

18IV. *Given functions $F_i(x_1, \cdots, x_n)$ and $G_i(x_1, \cdots, x_n)$ $(i = 1, \cdots, m)$ which are defined for all n-tuples of positive integers $(x_1, \cdots, x_n)$ and whose values are positive integers, if F_i and G_i are definable formally, then there can be found a formula $\boldsymbol{L}$ such that* (a) *if solutions of the system of equations*

$$(3) \qquad F_i(x_1, \cdots, x_n) = G_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, m)$$

exist, $\boldsymbol{L}$ enumerates them formally, and* (b) *if less than k different solutions exist, $\boldsymbol{L}(\boldsymbol{k})$ does not have a normal form.*

For example, a formula $\mathfrak{F}$ can be found such that (a) $\mathfrak{F}$ enumerates the solutions of $x^t + y^t = z^t$ $(t > 2)$ in positive integers, if such solutions exist, and (b) the Fermat problem is equivalent to the problem of whether $\mathfrak{F}(1)$ has a normal form.

We have noted that a theory of formal definition of functions of natural numbers, similar to our theory for functions of positive integers, can be constructed. It is also easy to construct a like theory for integers, if the integer x is represented by the formula $[\boldsymbol{x}_1, \boldsymbol{x}_2]$, where x_1, x_2 are the least positive integers such that $x_1 - x_2 = x$; and a like theory for rational numbers, if the rational number x is represented by the formula $[\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3]$ where x_1, x_2, x_3 are the least positive integers such that $(x_1 - x_2)/x_3 = x$. In particular, theorems corresponding to 18IV can be proved for each of these theories.

Given any formula T in the notation of *Principia Mathematica,* there can be found a well-formed expression $\boldsymbol{K}$ such that the problem whether T is provable in the system of *Principia* is equivalent to the problem, whether $\boldsymbol{K}$ has a normal form. Indeed, suppose we have given any formula T and any system of formal logic F, for which the condition is satisfied that there is a

* That is, there is an enumeration of the solutions as $(x_{j_1}, \cdots, x_{j_n})$ $(j = 1, 2, \cdots)$ such that $L(j)$ conv $[\boldsymbol{x}_{j_1}, \cdots, \boldsymbol{x}_{j_n}]$, in the notation of § 8.

class M of formulas such that (a) all provable formulas of F belong to M, (b) T belongs to M, and (c) there exists a one-to-one correspondence of M to a class of positive integers such that the numbers corresponding to provable formulas are enumerable formally in the sense of the present theory (let t correspond to T, and $\boldsymbol{L}$ enumerate the numbers ordered to provable formulas). Then the problem whether T is provable in F is equivalent to the problem whether $\mathfrak{P}(\lambda n \cdot \delta_t^{L(n)}, 1)$ has a normal form.

19. A representation of the logic C_1 within itself. Let C_1 denote the logic whose formal axioms are 1, 3-11, 14-16, and whose rules of procedure are I-V.

The objective of this section is its last theorem, to establish which we utilize a representation of the logic C_1 within itself in the fashion of Gödel.* Our particular choice of a representation serves to simplify the formal proofs. Instead of setting it up directly, we first set up a representation of the combinations without free symbols by formulas which will be called "metads," and then avail ourselves of a relation suggested by Rosser between C_1 and a certain system of combinations without free symbols.

Let $\mathfrak{r}$ be an expression such that $\mathfrak{r}(1)$ conv $\lambda m \cdot m(\lambda pq \cdot I^q(p))$ and $\mathfrak{r}(S(\boldsymbol{k}))$ conv $\lambda m \cdot m(\lambda pqr \cdot I^{\mathfrak{r}(\boldsymbol{k},q)}(I^{\mathfrak{r}(\boldsymbol{k},r)}(p)))$, and $\mathfrak{h}$ an expression such that $\mathfrak{h}(1)$ conv $\lambda p \cdot \mathfrak{r}(1, \lambda m \cdot m(1, p))$ and $\mathfrak{h}(S(\boldsymbol{k}))$ conv $\lambda pq \cdot \mathfrak{r}(S(\boldsymbol{k}), \lambda m \cdot m(S(\boldsymbol{k}), p, q))$ $(k = 1,2,3,\cdots)$.† Abbreviate $\boldsymbol{a}(\mathfrak{h})$ to $|\, \boldsymbol{a} \,|$, $\{\lambda xm \cdot m(1,x)\}(\boldsymbol{x})$ to $[\boldsymbol{x}]$, $\{\lambda abm \cdot m(S(|\, a \,|), a, b)\}(\boldsymbol{a}, \boldsymbol{b})$ to $[\boldsymbol{a}, \boldsymbol{b}]$, and $[[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}], [\boldsymbol{x}_{2^{r-1}+1}, \cdots, \boldsymbol{x}_{2^r}]]$ to $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^r}]$. A formula $\boldsymbol{a}$ shall be called a *metad* (of *rank* r) if $\boldsymbol{a}$ conv $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$, where $x_1, \cdots, x_{2^{r-1}}$ is a set of 1's and 2's.

19I. *If $\boldsymbol{a}$ is a metad of rank r, then $|\, \boldsymbol{a} \,|$ conv $\boldsymbol{r}$.*

For, by induction with respect to r, if $\boldsymbol{a}$ is a metad of rank r, then $\mathfrak{r}(\boldsymbol{r}, \boldsymbol{a})$ conv $\boldsymbol{r}$ and $|\, \boldsymbol{a} \,|$ conv $\mathfrak{r}(\boldsymbol{r}, \boldsymbol{a})$ (cf. the proof of 19.1).

Let ad $\rightarrow \lambda\mu \cdot [\phi([1])\phi([2]) \cdot [\phi(a)\phi(b)\, |\, a \,| = |\, b \,|] \supset_{ab} \phi([a, b])] \supset_\phi \phi(\mu)$.

19.1(x): $\quad$ ad$([\boldsymbol{x}])$ $\quad (x = 1, 2)$.

* *Loc. cit.*

† Henceforth the introduction of expressions in accordance with the Theorems 15III-15V will be made in an abbreviated manner, as here where $\mathfrak{r}$ is supposed to satisfy not only the stated relations but also the relation $N(K) \vdash' \mathfrak{r}(S(K)) =' \lambda m \cdot m(\lambda pqr \cdot \mathfrak{r}(K, q, I, \mathfrak{r}(K, r, I, p)))$ (cf. 15IV), and $\mathfrak{h}$ is supposed to satisfy not only the stated relations but also the relation $N(K) \vdash' \mathfrak{h}(S(K)) =' \lambda pq \cdot \mathfrak{r}(S(K), \lambda m \cdot m(S(K), p, q))$ (cf. 15III).

19.2: $$\mathrm{ad}(a) \supset_a N(|a|).$$

Proofs. (1) $N(|[\boldsymbol{x}]|) \cdot |[\boldsymbol{x}]| = \mathfrak{r}(|[\boldsymbol{x}]|, [\boldsymbol{x}])$ $(x = 1, 2)$ is provable by conversion from $N(1) \cdot 1 = 1$. (2) Assume $N(|a|)$, $|a| = \mathfrak{r}(|a|, a)$, $N(|b|)$, $|b| = \mathfrak{r}(|b|, b)$, $|a| = |b|$. Then $|[a, b]|$ conv $\mathfrak{h}(S(|a|), a, b)$, $= \{\lambda pq \cdot \mathfrak{r}(S(|a|), \lambda m \cdot m(S(|a|), p, q))\}(a, b)$ (using the assumption $N(|a|)$ and the last property of $\mathfrak{h}$ as selected in accordance with 15III), conv $\mathfrak{r}(S(|a|), [a, b])$, $= \{\lambda m \cdot m(\lambda pqr \cdot I^{\mathfrak{r}(|a|,q)}(I^{\mathfrak{r}(|a|,r)}(p)))\}([a, b])$ (using $N(|a|)$ and the last property of $\mathfrak{r}$ as selected in accordance with 15IV), conv $I^{\mathfrak{r}(|a|,a)}(I^{\mathfrak{r}(|a|,b)}(S(|a|)))$, $= I^{\mathfrak{r}(|a|,a)}(I^{\mathfrak{r}(|b|,b)}(S(|a|)))$ (by the assumption $|a| = |b|$), $= I^{|a|}(I^{|b|}(S(|a|))$ (by the assumptions $|a| = \mathfrak{r}(|a|, a)$ and $|b| = \mathfrak{r}(|b|, b)$), $= I(I(S(|a|)))$ $(N(|a|), N(|b|), 7.2)$, conv $S(|a|)$. Hence $|[a, b]| = \mathfrak{r}(|[a, b]|, [a, b])$ (note the occurrence of $\mathfrak{r}(S(|a|), [a, b])$ in the foregoing chain of equalities) and $N(|[a, b]|)$ (using $N(|a|)$ and 3.2). (3) Hence, if $\mathfrak{D}_\phi \rightarrow \phi([1])\phi([2]) \cdot [\phi(a)\phi(b)\ |a| = |b|] \supset_{ab} \phi([a, b])$, $\{\lambda\phi \cdot \mathfrak{D}_\phi\}(\lambda\alpha \cdot N(|\alpha|) \cdot |\alpha| = \mathfrak{r}(|\alpha|, \alpha))$ is provable. Hence $\Sigma\phi \cdot \mathfrak{D}_\phi$. $\mathfrak{D}_\phi \vdash \phi([\boldsymbol{x}])$ $(x = 1, 2)$. By Theorem I, $\mathrm{ad}([\boldsymbol{x}])$. (4) Now $\Sigma a \cdot \mathrm{ad}(a)$. Assume $\mathrm{ad}(a)$. From $\mathrm{ad}(a)$ and $\{\lambda\phi \cdot \mathfrak{D}_\phi\}(\lambda\alpha \cdot N(|\alpha|) \cdot |\alpha| = \mathfrak{r}(|\alpha|, \alpha))$ by Rule V, $\{\lambda\phi \cdot \phi(a)\}(\lambda\alpha \cdot N(|\alpha|) \cdot |\alpha| = \mathfrak{r}(|\alpha|, \alpha))$. Thence, $N(|a|)$.

19.3: $[\mathrm{ad}(a)\mathrm{ad}(b)\ |a| = |b|] \supset_{ab} \mathrm{ad}([a, b])$.

19.4: $[\phi([1])\phi([2])$
$\cdot [\mathrm{ad}(a)\phi(a)\mathrm{ad}(b)\phi(b)\ |a| = |b|] \supset_{ab} \phi([a, b])] \supset_\phi \cdot \mathrm{ad}(c) \supset_c \phi(c)$.

These theorems follow from 19.1 and the formula $\Sigma\phi \cdot \mathfrak{D}_\phi$ occurring in the proof of 19.1 in the same manner as 3.2 and 3.3 from 3.1 and $\mathfrak{A}_5$ of the proof of 3.1. The inference of an expression of the form $\mathrm{ad}(\boldsymbol{c}) \supset_{\boldsymbol{c}} \boldsymbol{F}(\boldsymbol{c})$ by means of 19.4 will be said to be by *induction* (*with respect to* $\boldsymbol{c}$).

Choose $\mathfrak{m}_j$ so that $\mathfrak{m}_j(1)$ conv I, $\mathfrak{m}_1(S(\boldsymbol{k}))$ conv $\lambda m \cdot m(\lambda pqr \cdot I^p(I^{|r|}(q)))$, $\mathfrak{m}_2(S(\boldsymbol{k}))$ conv $\lambda m \cdot m(\lambda pqr \cdot I^p(I^{|q|}(r)))$, and let $\mathfrak{M}_j \rightarrow \lambda\rho \cdot \mathfrak{m}_j(|\rho|, \rho)$ $(j = 1, 2;\ k = 1, 2, 3, \cdots)$.* We abbreviate $\mathfrak{M}_j(\boldsymbol{a})$ to $\boldsymbol{a}_j$, $\mathfrak{M}_i(\mathfrak{M}_j(\boldsymbol{a}))$ to $\boldsymbol{a}_{ji}$, etc., when $\boldsymbol{a}$ is a metad or represents a metad in the formal argument.

19II. *If* $\boldsymbol{a}$ *conv* $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$, *where* $x_1, \cdots, x_{2^{r-1}}$ *are* 1's *and* 2's *and* $r > 1$, *then* $\boldsymbol{a}_1$ *conv* $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-2}}]$ *and* $\boldsymbol{a}_2$ *conv* $[\boldsymbol{x}_{2^{r-2}+1}, \cdots, \boldsymbol{x}_{2^{r-1}}]$.

19.5: $[\mathrm{ad}(a)\mathrm{ad}(b)\,|a| = |b|] \supset_{ab} \cdot |[a, b]| = S(|a|) \cdot [a, b]_1 = a \cdot [a, b]_2 = b$.

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* We know that there exists an expression $\mathfrak{m}_1$ having the specified properties, and the property $N(\boldsymbol{K}) \vdash' \mathfrak{m}_1(S(\boldsymbol{K})) =' \lambda m \cdot m(\lambda pqr \cdot I^p(I^{|r|}(q)))$, by use of 15III in conjunction with 15Ie and 7.2, taking for $\boldsymbol{F}$ the expression $\lambda x \cdot I^x(\lambda m \cdot m(\lambda pqr \cdot I^p(I^{|r|}(q))))$. The introduction of $\mathfrak{m}_2$ is justified in the same manner.

Proof. Assume $\mathrm{ad}(a)\ \mathrm{ad}(b) | a | = | b |$. By (4) of the proof of 19.1 and 19.2, we can infer $N(| a |), | a | = \mathfrak{r}(| a |, a), N(| b |), | b | = \mathfrak{r}(| b |, b)$, and hence, by (2) of the same proof, $|[a, b]| = S(| a |)$. Then also $[a, b]_1$ conv $\mathfrak{m}_1(|[a, b]|, [a, b]), = \mathfrak{m}_1(S(| a |), [a, b]), = \{\lambda m \cdot m(\lambda pqr \cdot I^p(I^{|r|}(q)))\}([a, b])$ (by a supposition concerning $\mathfrak{m}_1$), conv $I^{S(|a|)}(I^{|b|}(a))$, $= I(I(a))$ (19.2, 7.2), conv a. $\mathrm{ad}(a) \vdash E(a)$. Hence, by § 2, $[a, b]_1 = a$. Similarly $[a, b]_2 = b$.

Let $\mathfrak{c} \to \lambda m \cdot m(\lambda p \cdot I^p)$.

19.6: $[\mathrm{ad}(a) | a | = 1] \supset_a \cdot \mathfrak{c}(a) < 3 \cdot a = [\mathfrak{c}(a)]$.

19.7: $[\mathrm{ad}(a) | a | > 1] \supset_a \cdot a = [a_1, a_2] \cdot ad(a_1) \cdot ad(a_2) \cdot | a_1 | = | a_2 | = | a | - 1$.

Proofs. Choose an expression $\mathfrak{B}$ such that $\mathfrak{B}(1)$ conv $\lambda a \cdot \mathfrak{c}(a) < 3 \cdot a = [\mathfrak{c}(a)] \cdot E(I)$ and $\mathfrak{B}(2)$ conv $\lambda a \cdot a = [a_1, a_2] \cdot \mathrm{ad}(a_1) \cdot \mathrm{ad}(a_2) \cdot | a_1 | = | a_2 | = | a | - 1$. Then, using 19.5, the lemma $\mathrm{ad}(a) \supset_a \mathfrak{B}(\epsilon_1{}^{|a|}, a)$ can be proved by induction. 19.6 and 19.7 follow.

19.8: $\mathrm{ad}(a) \supset_a \cdot \mathrm{ad}(a_1) ad(a_2)$.

Proof. Assume $\mathrm{ad}(a)$. Case 1: $\epsilon_1{}^{|a|} = 1$. Then $| a | = 1$; and, using the definitions of $\mathfrak{M}_j$ and $\mathfrak{m}_j$, $a = a_j$ $(j = 1, 2)$. Using the latter, $\mathrm{ad}(a_j)$ follows from $\mathrm{ad}(a)$. Case 2: $\epsilon_1{}^{|a|} = 2$. Then $| a | > 1$; and $\mathrm{ad}(a_1)\mathrm{ad}(a_2)$ can be proved by means of 19.7.

In the remainder of this paper, we shall mean by a *combination* a combination, in the sense of § C6, which contains no free symbols; in other words, a combination whose terms are I's and J's. If $\boldsymbol{T}$ is the only term of a combination, the *rank* of $\boldsymbol{T}$ shall be 1; if $\boldsymbol{T}$ is a term of $\boldsymbol{M}$ of rank r, then the rank of $\boldsymbol{T}$ as a term of $\{\boldsymbol{M}\}(\boldsymbol{N})$ or $\{\boldsymbol{N}\}(\boldsymbol{M})$ shall be $r + 1$. The *rank* of a combination shall be the rank of its term of highest rank. A combination shall be *uniform* if all its terms have the same rank. (A uniform combination $\boldsymbol{A}'$ of rank r has 2^{r-1} terms, and they occur in $\boldsymbol{A}'$ in a linear series—cf. C6III). A uniform combination $\boldsymbol{A}'$ shall *represent* a combination $\boldsymbol{A}$, if $\boldsymbol{A}'$ is derivable from $\boldsymbol{A}$ by zero or more substitutions of $I(\boldsymbol{T})$ for $\boldsymbol{T}$, where $\boldsymbol{T}$ is a term. Given the correspondence $\binom{I\ J}{1\ 2}$, $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$ shall *correspond* to a uniform combination $\boldsymbol{A}'$ if $x_1, \cdots, x_{2^{r-1}}$ is the series of the numbers which correspond to the respective terms of $\boldsymbol{A}'$. If $\boldsymbol{A}$ is a combination and $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$ corresponds to a uniform combination $\boldsymbol{A}'$ which represents $\boldsymbol{A}$, we write "$[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$." A metad $\boldsymbol{a}$ shall *represent* a combination $\boldsymbol{A}$ if $\boldsymbol{a}$ conv $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$ where $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$.

19III. *Suppose that* x_i, y_i $(i = 1, \cdots, 2^{r-1})$ *are* 1's *and* 2's. a. *Given a combination* $\boldsymbol{A}$, *a representing metad* $\boldsymbol{a}$ *can be found.* b. *If the metad* $\boldsymbol{a}$ *represents the combination* $\boldsymbol{A}$, *then* $\boldsymbol{a}$ *is of rank* $\geq$ *the rank of* $\boldsymbol{A}$. c. *If* $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$ *and* $[\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2_{r-1}}] \sim \boldsymbol{B}$, *then* $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}] \sim \{\boldsymbol{A}\}(\boldsymbol{B})$. d. *If* $[\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$, *then* $[1, \boldsymbol{x}_1, 1, \boldsymbol{x}_2, \cdots, 1, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$. e. *If both* $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$ and $[\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}] \sim \boldsymbol{A}$, *then* $x_i = y_i$. f. *If the metad* $\boldsymbol{a}$ *represents the combination* $\{\boldsymbol{F}\}(\boldsymbol{P})$, *then* $\boldsymbol{a}_1$ *represents* $\boldsymbol{F}$, *and* $\boldsymbol{a}_2$ *represents* $\boldsymbol{P}$.

Let $\mathfrak{e}$ be an expression such that $\mathfrak{e}(1)$ conv $\lambda p \cdot [[1], p]$ and $\mathfrak{e}(S(\boldsymbol{k}))$ conv $\lambda p \cdot [\mathfrak{e}(\boldsymbol{k}, p_1), \mathfrak{e}(\boldsymbol{k}, p_2)]$ $(k = 1, 2, \cdots)$. Let $\mathfrak{E} \rightarrow \lambda\rho \cdot \mathfrak{e}(|\,\rho\,|, \rho)$.

19IV. *If* $\boldsymbol{a}$ *conv* $[\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_{2^{r-1}}]$ $(x_1, \cdots, x_{2^{r-1}}$ *being* 1's *and* 2's), *then* $\mathfrak{E}(\boldsymbol{a})$ *conv* $[1, \boldsymbol{x}_1, 1, \boldsymbol{x}_2, \cdots, 1, \boldsymbol{x}_{2^{r-1}}]$.

The proof is by induction with respect to r (using 19I and 19II).

19.9: $$N(\rho) \supset_\rho \cdot \mathrm{ad}(a) \supset_a \cdot \mathrm{ad}(\mathfrak{E}^\rho(a)) \cdot |\,\mathfrak{E}^\rho(a)\,| = \rho + |\,a\,|.$$

Proof. $\mathrm{ad}(a) \supset_a \cdot \mathrm{ad}(\mathfrak{E}(a)) \cdot |\,\mathfrak{E}(a)\,| = S(|\,a\,|)$ is provable by induction with respect to a (using 19.1-19.3, 19.5), and 19.9 follows by induction with respect to ρ.

19.10: $$[N(\rho)\mathrm{ad}(a)\mathrm{ad}(b) \cdot \mathfrak{E}^\rho(a) = \mathfrak{E}^\rho(b)] \supset_{\rho a b} \cdot a = b.$$

Proof. Let $\mathfrak{e}'$ be an expression such that $\mathfrak{e}'(1)$ conv $\lambda p \cdot p_2$ and $\mathfrak{e}'(S(\boldsymbol{k}))$ conv $\lambda p \cdot [\mathfrak{e}'(\boldsymbol{k}, p_1), \mathfrak{e}'(\boldsymbol{k}, p_2)]$ $(k = 1, 2, \cdots)$. Let $\mathfrak{E}' \rightarrow \lambda\rho \cdot \mathfrak{e}'(|\,\rho\,| - 1, \rho)$. Then $\mathrm{ad}(a) \supset_a \cdot \mathfrak{E}'(\mathfrak{E}(a)) = a$ is provable by induction with respect to a, and $N(\rho) \supset_\rho \cdot \mathrm{ad}(a) \supset_a \cdot \mathfrak{E}'^\rho(\mathfrak{E}^\rho(a)) = a$ follows by induction with respect to ρ. 19.10 follows from the latter in the same manner as 11.4 from 11.2.

Let $\langle \boldsymbol{a}, \boldsymbol{b} \rangle \rightarrow [\mathfrak{E}^{|\boldsymbol{b}|}(\boldsymbol{a}), \mathfrak{E}^{|\boldsymbol{a}|}(\boldsymbol{b})]$.

19V. *If the metads* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *represent the combinations* $\boldsymbol{A}$ *and* $\boldsymbol{B}$, *respectively, then* $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ *is a metad which represents* $\{\boldsymbol{A}\}(\boldsymbol{B})$.

This follows from 19I, 15Id, 19IV, 19IIId, c.

19.11: $$\mathrm{ad}(a)\mathrm{ad}(b) \supset_{ab} \cdot \mathrm{ad}(\langle a, b \rangle).$$

Let $\mathfrak{D}$ be an expression such that $\mathfrak{D}(1)$ conv $\lambda pq \cdot \delta(p(\lambda n \cdot I^n), q(\lambda n \cdot I^n))$ and $\mathfrak{D}(S(\boldsymbol{k}))$ conv $\lambda pq \cdot \mathfrak{D}(\boldsymbol{k}, p_1, q_1) \circ \mathfrak{D}(\boldsymbol{k}, p_2, q_2)$ $(k = 1, 2, \cdots)$. Let $\Delta \rightarrow \lambda pq \cdot \mathfrak{D}(|\,p\,| + |\,q\,|, \mathfrak{E}^{|q|}(p), \mathfrak{E}^{|p|}(q))$, and abbreviate $\Delta(\boldsymbol{a}, \boldsymbol{b})$ to $\Delta^{\boldsymbol{a}}_{\boldsymbol{b}}$.

19VI. *If the metads* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *both represent the combination* $\boldsymbol{A}$, *then* $\Delta^{\boldsymbol{a}}_{\boldsymbol{b}}$ *conv* 2.

Proof. By induction with respect to r (using 15Ie, 19II), if $x_1, \cdots, y_{2^{r-1}}$ are 1's and 2's, $\mathfrak{D}(\boldsymbol{r}, [\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}], [\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}])$ conv $\delta^{\boldsymbol{x}_1}_{\boldsymbol{y}_1} \circ \cdots \circ \delta^{\boldsymbol{x}_{2^{r-1}}}_{\boldsymbol{y}_{2^{r-1}}}$. Hence, by 15Il, j and 19IIIe, if $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$ and $[\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}] \sim \boldsymbol{A}$, then $\mathfrak{D}(\boldsymbol{r}, [\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}], [\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}])$ conv 2. Moreover, by 19IV, 19I, 15Id and 19IIId, if $\boldsymbol{a}$ conv $[\boldsymbol{x}'_1, \cdots, \boldsymbol{x}'_{2^{m-1}}]$, $[\boldsymbol{x}'_1, \cdots, \boldsymbol{x}'_{2^{m-1}}] \sim \boldsymbol{A}$, $\boldsymbol{b}$ conv $[\boldsymbol{y}'_1, \cdots, \boldsymbol{y}'_{2^{n-1}}]$, $[\boldsymbol{y}'_1, \cdots, \boldsymbol{y}'_{2^{n-1}}] \sim \boldsymbol{A}$, then there are $x_1, \cdots, x_{2^{r-1}}$, $y_1, \cdots, y_{2^{r-1}}$ $(r = m + n)$ such that $\mathfrak{E}^{|\boldsymbol{b}|}(\boldsymbol{a})$ conv $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$, $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}] \sim \boldsymbol{A}$, $\mathfrak{E}^{|\boldsymbol{a}|}(\boldsymbol{b})$ conv $[\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}]$, $[\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{2^{r-1}}] \sim \boldsymbol{A}$.

19.12: $\qquad \mathrm{ad}(a)\mathrm{ad}(b) \supset_{ab} \cdot M(\Delta_b{}^a)$.

19.13: $\qquad \mathrm{ad}(a)\mathrm{ad}(b) \supset_{ab} \cdot \Delta_b{}^a = \Delta_a{}^b$.

19.14: $\qquad \mathrm{ad}(a)\mathrm{ad}(b) \supset_{ab} \cdot N(\rho) \supset_\rho \cdot \Delta(a, \mathfrak{E}^\rho(b)) = \Delta_b{}^a$.

Proofs. If $\mathfrak{B} \rightarrow \lambda a \cdot [\mathrm{ad}(b) \mid a \mid = \mid b \mid] \supset_b \cdot M(\mathfrak{D}(\mid a \mid, a, b)) \cdot \mathfrak{D}(\mid a \mid, a, b) = \mathfrak{D}(\mid a \mid, b, a) \cdot \mathfrak{D}(S(\mid a \mid), \mathfrak{E}(a), \mathfrak{E}(b)) = \mathfrak{D}(\mid a \mid, a, b)$, the lemma $\mathrm{ad}(a) \supset_a \mathfrak{B}(a)$ can be proved by induction, using first 19.6, 14.10, 14.11, 14.5, and then the relation $\mathrm{ad}(l)$, $\mathrm{ad}(m)$, $\mid l \mid = \mid m \mid$, $\mathrm{ad}(b)$, $|[l, m]| = \mid b \mid$ $\vdash \mathfrak{D}(|[l, m]|, [l, m], b) = \mathfrak{D}(\mid l \mid, l, b_1) \circ \mathfrak{D}(\mid m \mid, m, b_2) \cdot \mathfrak{D}(|[l, m]|, b, [l, m]) = \mathfrak{D}(\mid l \mid, b_1, l) \circ \mathfrak{D}(\mid m \mid, b_2, m) \cdot \mathfrak{D}(S(|[l, m]|), \mathfrak{E}([l, m]), \mathfrak{E}(b)) = \mathfrak{D}(S(\mid l \mid), \mathfrak{E}(l), \mathfrak{E}(b_1)) \circ \mathfrak{D}(S(\mid m \mid), \mathfrak{E}(m), \mathfrak{E}(b_2))$ (which follows from 19.2, 19.5, 19.9, 19.7), and 19.2, 19.3, 19.5, 19.7, 14.2. 19.12-19.14 follow from the lemma, 19.2, 19.9, and the relation $\mathrm{ad}(a)\mathrm{ad}(b) \vdash \Delta(a, \mathfrak{E}(b)) = \mathfrak{D}(S(\mid \mathfrak{E}^{|b|}(a) \mid), \mathfrak{E}(\mathfrak{E}^{|b|}(a)), \mathfrak{E}(\mathfrak{E}^{|a|}(b)))$.

19.15: $\qquad [\mathrm{ad}(a)\mathrm{ad}(b) \cdot \mid a \mid = \mid b \mid \cdot \Delta_b{}^a = 2] \supset_{ab} \cdot a = b$.

19.16: $\qquad \mathrm{ad}(a) \supset_a \cdot \Delta_a{}^a = 2$.

Proofs. If $\mathfrak{C} \rightarrow \lambda a \cdot [\mathfrak{D}(\mid a \mid, a, a) = 2] \cdot [\mathrm{ad}(b) \cdot \mid a \mid = \mid b \mid \cdot \mathfrak{D}(\mid a \mid, a, b) = 2] \supset_b \cdot a = b$, then $\mathrm{ad}(a) \supset_a \mathfrak{C}(a)$ is provable by induction, using first 19.6, 14.14, and then $\mathrm{ad}(a) \supset_a \mathfrak{B}(a)$, 19.2, 19.3, 19.5, 19.7, 14.6 and the relation $\mathrm{ad}(l)$, $\mathrm{ad}(m)$, $\mid l \mid = \mid m \mid$, $\mathrm{ad}(b)$, $|[l, m]| = \mid b \mid \vdash \mathfrak{D}(|[l, m]|, [l, m], b) = \mathfrak{D}(\mid l \mid, l, b_1) \circ \mathfrak{D}(\mid m \mid, m, b_2)$. 19.15 and 19.16 follow, using 19.10.

Let $\mathfrak{J}$ be an expression such that $\mathfrak{J}(1)$ conv I and $\mathfrak{J}(2)$ conv J, and $\mathfrak{g}$ an expression such that $\mathfrak{g}(1)$ conv $\lambda a \cdot a(\lambda pq \cdot I^p(\mathfrak{J}(q)))$ and $\mathfrak{g}(S(\boldsymbol{k}))$ conv $\lambda a \cdot \mathfrak{g}(\boldsymbol{k}, a_1, \mathfrak{g}(\boldsymbol{k}, a_2))$ $(k = 1, 2, \cdots)$. Let $\mathfrak{G} \rightarrow \lambda a \cdot \mathfrak{g}(\mid a \mid, a)$.

19VII. *If the metad* $\boldsymbol{a}$ *represents the combination* $\boldsymbol{A}$, *then* $\mathfrak{G}(\boldsymbol{a})$ *conv* $\boldsymbol{A}$.

For, by induction with respect to r, if $[\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}]$ corresponds to a uniform combination $\boldsymbol{A}'$, $\mathfrak{G}([\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2^{r-1}}])$ conv $\boldsymbol{A}'$. If $\boldsymbol{A}'$ represents $\boldsymbol{A}$, $\boldsymbol{A}'$ conv $\boldsymbol{A}$.

Let $\mathfrak{i}$ be an expression such that $\mathfrak{i}(1)$ conv $[1]$ and $\mathfrak{i}(S(\boldsymbol{k}))$ conv $[\mathfrak{i}(\boldsymbol{k}), \mathfrak{i}(\boldsymbol{k})]$ $(k = 1, 2, \cdots)$.

19.17: $\quad N(r) \supset_r \cdot \operatorname{ad}(\mathfrak{i}(r)) \cdot |\,\mathfrak{i}(r)\,| = r \cdot \mathfrak{G}(\mathfrak{i}(r)) = I.$

19.18: $\quad [\operatorname{ad}(a) \cdot |\,a\,| > 1 \cdot E(\mathfrak{G}(a))] \supset_a \cdot \mathfrak{G}(a) = \mathfrak{G}(a_1, \mathfrak{G}(a_2)).$

Proofs. 19.17 is provable by induction with respect to r. 19.17 $\vdash \Sigma a \cdot \operatorname{ad}(a) \cdot |\,a\,| > 1 \cdot E(\mathfrak{G}(a))$; and, assuming $\operatorname{ad}(a) \cdot |\,a\,| > 1 \cdot E(\mathfrak{G}(a))$, $\mathfrak{G}(a) = \mathfrak{G}(a_1, \mathfrak{G}(a_2))$ (by 19.7, 12.5, § 2). Hence, by Theorem I, $\vdash$ 19.18.

19.19: $\quad \begin{array}{l} N(\rho) \supset_\rho \cdot \operatorname{ad}(a) E(\mathfrak{G}(a)) \supset_a \cdot \mathfrak{G}(a) = \mathfrak{G}(\mathfrak{C}^\rho(a)). \\ N(\rho) \supset_\rho \cdot \operatorname{ad}(a) E(\mathfrak{G}(\mathfrak{C}^\rho(a))) \supset_a \cdot \mathfrak{G}(a) = \mathfrak{G}(\mathfrak{C}^\rho(a)). \end{array}$

Proof. Note that $N(\boldsymbol{n})$, 19.17 $\vdash \Sigma \boldsymbol{a} \cdot \operatorname{ad}(\boldsymbol{a}) \cdot |\,\boldsymbol{a}\,| = \boldsymbol{n} \cdot E(\mathfrak{G}(\boldsymbol{a}))$. Using this relation, 19.1, 19.9, 19.5, 19.7, 11.2, §2, and Theorem I, we can prove $N(r) \supset_r \cdot [\operatorname{ad}(a) \cdot |\,a\,| = r \cdot E(\mathfrak{G}(a))] \supset_a \cdot \mathfrak{G}(a) = \mathfrak{G}(\mathfrak{C}(a))$ by induction with respect to r. Thence, using 19.17, 19.2 and Theorem I, $\operatorname{ad}(a) E(\mathfrak{G}(a)) \supset_a \cdot \mathfrak{G}(a) = \mathfrak{G}(\mathfrak{C}(a))$. The first of the formulas 19.19 follows by induction with respect to ρ; and the second is proved similarly.

19.20: $\quad \operatorname{ad}(a) \operatorname{ad}(b) E(\mathfrak{G}(a, \mathfrak{G}(b))) \supset_{ab} \cdot \mathfrak{G}(a, \mathfrak{G}(b)) = \mathfrak{G}(\langle a, b \rangle).$

Proof. 19.17 $\vdash \Sigma ab \cdot \operatorname{ad}(a) \operatorname{ad}(b) E(\mathfrak{G}(a, \mathfrak{G}(b)))$. Assume $\operatorname{ad}(a) \operatorname{ad}(b) E(\mathfrak{G}(a, \mathfrak{G}(b)))$. Then $\mathfrak{G}(a, \mathfrak{G}(b)) = \mathfrak{G}(\mathfrak{C}^{|b|}(a), \mathfrak{G}(\mathfrak{C}^{|a|}(b)))$ (19.19, 19.2), $= \mathfrak{G}(\langle a, b \rangle)$ (19.2, 19.9, 19.5, def. of $\mathfrak{G}$).

19.21: $\quad [\operatorname{ad}(a) \operatorname{ad}(b) E(\mathfrak{G}(a)) \cdot \Delta_b{}^a = 2] \supset_{ab} \cdot \mathfrak{G}(a) = \mathfrak{G}(b).$

Proof. 19.17, 19.16 $\vdash \Sigma ab \cdot \operatorname{ad}(a) \operatorname{ad}(b) E(\mathfrak{G}(a)) \cdot \Delta_b{}^a = 2$. Assuming $\operatorname{ad}(a) \operatorname{ad}(b) E(\mathfrak{G}(a)) \cdot \Delta_b{}^a = 2$, then $\mathfrak{G}(a) = \mathfrak{G}(\mathfrak{C}^{|b|}(a))$ (19.19, 19.2), $= \mathfrak{G}(\mathfrak{C}^{|a|}(b))$ (19.15, 19.14, 19.13, 19.2, 19.9), $= \mathfrak{G}(b)$.

A combination $\bar{\boldsymbol{A}}$ shall be said to be *representative* of a formula $\boldsymbol{A}$, if $\bar{\boldsymbol{A}}$ conv $\lambda \Pi \Sigma \& \cdot \boldsymbol{A} \cdot E(\Pi)$.

19VIII. *Given a formula* $\boldsymbol{A}$ *having no free symbols other than* Π, Σ *and* &, *a representative combination* $\bar{\boldsymbol{A}}$ *can be found.*

Proof. By C6V, there is a combination $\bar{\boldsymbol{A}}$ (in the sense of § C6) such that $\bar{\boldsymbol{A}}$ conv $\lambda \Pi \Sigma \& \cdot \boldsymbol{A} \cdot E(\Pi)$. Under the hypothesis, $\lambda \Pi \Sigma \& \cdot \boldsymbol{A} \cdot E(\Pi)$ con-

tains no free symbols, and hence, by C5VI, $\bar{\boldsymbol{A}}$ is a combination in the present sense.

Let the subsequences (including the null sequence) of the sequence $\Pi, \Sigma, \&$ be $X_{i1}, \cdots, X_{ia_i}$ $(i = 1, \cdots, 2^3)$. By C6V and C5VI, there are combinations $\mathfrak{S}_{ij}$, $\mathfrak{T}_i$ and $\mathfrak{U}_{ij}$ convertible into $\lambda fp\Pi\Sigma\& \cdot f(X_{i1}, \cdots, X_{ia_i}, p(X_{j1}, \cdots, X_{ja_j})) \cdot E(\Pi)$, $\lambda f\Pi\Sigma\& \cdot \Sigma(f(X_{i1}, \cdots, X_{ia_i})) \cdot E(\Pi)$ and $\lambda fg\Pi\Sigma\& \cdot \Pi(f(X_{i1}, \cdots, X_{ia_i}), g(X_{j1}, \cdots, X_{ja_j})) \cdot E(\Pi)$, respectively $(i, j = 1, \cdots, 2^3)$.*

We denote the rules of procedure of Rosser, *loc. cit.*, Section H,† by $R_1, \cdots, R_{38}$, and list the rules R_{ik}, "*If* $\mathfrak{S}_{ik}(\boldsymbol{f}, \boldsymbol{p})$, *then* $\mathfrak{T}_i(\boldsymbol{f})$," $(i, k = 1, \cdots, 2^3)$, as R_{39}-R_{102}, and the rules R_{ijk}, "*If* $\mathfrak{U}_{ij}(\boldsymbol{f}, \boldsymbol{g})$ *and* $\mathfrak{S}_{ik}(\boldsymbol{f}, \boldsymbol{p})$, *then* $\mathfrak{S}_{jk}(\boldsymbol{g}, \boldsymbol{p})$," $(i, j, k = 1, \cdots, 2^3)$ as R_{103}-R_{614}.

19IX(t). *If* $\boldsymbol{C}$ *is derivable from* $\boldsymbol{A}$ *(*$\boldsymbol{A}$ *and* $\boldsymbol{B}$*) by an application of* R_t, *then* $\boldsymbol{A}(\Pi, \Sigma, \&)$ $(\boldsymbol{A}(\Pi, \Sigma, \&), \boldsymbol{B}(\Pi, \Sigma, \&)) \vdash \boldsymbol{C}(\Pi, \Sigma, \&)$, $(t = 1, \cdots, 614)$.

Proof. If $\boldsymbol{C}$ is derivable from $\boldsymbol{A}$ by an application of one of R_1-R_{38}, then $\boldsymbol{A}$ conv $\boldsymbol{C}$. If $\boldsymbol{C}$ is derivable from $\boldsymbol{A}$ ($\boldsymbol{A}$ and $\boldsymbol{B}$) by an application of one of the rules $R_{ik}(R_{ijk})$, then $\boldsymbol{C}(\Pi, \Sigma, \&)$ is derivable from $\boldsymbol{A}(\Pi, \Sigma, \&)$ $(\boldsymbol{A}(\Pi, \Sigma, \&)$ and $\boldsymbol{B}(\Pi, \Sigma, \&))$ by conversion, Rule IV (V) and the relations $\boldsymbol{PQ} \vdash \boldsymbol{P}$, $\boldsymbol{PQ} \vdash \boldsymbol{Q}$ and $\boldsymbol{P}, \boldsymbol{Q} \vdash \boldsymbol{PQ}$.

Let $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ be combinations representative of Axioms 1, 3-11, 14-16, respectively, and let $\mathfrak{a}_1, \cdots, \mathfrak{a}_{13}$ be metads representing $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$, respectively (cf. 19VIII, 19IIIa).

19X. *If the combination* $\bar{\boldsymbol{D}}$ *is representative of a formula* $\boldsymbol{D}$ *provable in* C_1, *then* $\bar{\boldsymbol{D}}$ *is derivable from* $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ *by means of Rules* R_1-R_{614}.

Proof. Under the hypothesis, $\bar{\boldsymbol{D}}$ is derivable from $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ by means of conversion and the two rules

IV′. *If* $\lambda\Pi\Sigma\& \cdot \boldsymbol{F}(\boldsymbol{P}) \cdot E(\Pi)$, *then* $\lambda\Pi\Sigma\& \cdot \Sigma(\boldsymbol{F}) \cdot E(\Pi)$.

V′. *If* $\lambda\Pi\Sigma\& \cdot \Pi(\boldsymbol{F}, \boldsymbol{G}) \cdot E(\Pi)$ *and* $\lambda\Pi\Sigma\& \cdot \boldsymbol{F}(\boldsymbol{P}) \cdot E(\Pi)$, *then* $\lambda\Pi\Sigma\& \cdot \boldsymbol{G}(\boldsymbol{P}) \cdot E(\Pi)$.

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* More explicitly, let $a_1 = 3$, $a_2 = a_3 = a_4 = 2$; $a_5 = a_6 = a_7 = 1$, $a_8 = 0$; and let X_{11}, X_{12}, X_{13}; X_{21}, X_{22}; X_{31}, X_{32}; X_{41}, X_{42}; X_{51}; X_{61}; X_{71} stand for $\Pi, \Sigma, \&$; Π, Σ; $\Pi, \&$; $\Sigma, \&$; Π; Σ; $\&$, respectively. Then $\mathfrak{S}_{13}$ shall be a combination convertible into $\lambda fp\Pi\Sigma\& \cdot f(\Pi, \Sigma, \&, p(\Pi, \&)) \cdot E(\Pi)$, $\mathfrak{S}_{85}$ a combination convertible into $\lambda fp\Pi\Sigma\& \cdot f(p(\Pi)) \cdot E(\Pi)$, etc.

† See the footnote of § C6 (*Annals of Mathematics*, vol. 35, p. 537, ([12])).

If $\lambda \Pi \Sigma \& \cdot \boldsymbol{F}(\boldsymbol{P}) \cdot E(\Pi)$ contains no free symbols, and if $X_{i1}, \cdots, X_{ia_i}$ and $X_{k1}, \cdots, X_{ka_k}$ are the sets of the symbols Π, Σ, & which occur in $\boldsymbol{F}$ and $\boldsymbol{P}$, respectively, as free symbols, then $\lambda X_{i1} \cdots X_{ia_i} \cdot \boldsymbol{F}$ and $\lambda X_{k1} \cdots X_{ka_k} \cdot \boldsymbol{P}$ contain no free symbols, and are hence convertible into combinations $\boldsymbol{F}'$ and $\boldsymbol{P}'$, respectively (C6V, C5VI). Then $\lambda \Pi \Sigma \& \cdot \boldsymbol{F}(\boldsymbol{P}) \cdot E(\Pi)$ conv $\mathfrak{S}_{ik}(\boldsymbol{F}', \boldsymbol{P}')$ and $\lambda \Pi \Sigma \& \cdot \Sigma(\boldsymbol{F}) \cdot E(\Pi)$ conv $\mathfrak{T}_i(\boldsymbol{F}')$. Hence, if $\boldsymbol{A}$ (containing no free symbols) yields $\boldsymbol{C}$ by an application of IV′, then $\boldsymbol{C}$ is derivable from $\boldsymbol{A}$ by conversion and an application of one of the rules R_{ik} in which the premise and conclusion are combinations. Similarly, if $\boldsymbol{A}$ and $\boldsymbol{B}$ (containing no free symbols) yield $\boldsymbol{C}$ by an application of V′, then $\boldsymbol{C}$ is derivable from $\boldsymbol{A}$ and $\boldsymbol{B}$ by conversion and an application of one of the rules R_{ijk} in which the premise and conclusion are combinations. The formulas derivable from $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ by conversion, IV′ and V′ contain no free symbols (cf. C5V Cor.). Hence $\bar{\boldsymbol{D}}$ is derivable from $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ by conversion and applications of R_{ik} and R_{ijk} in which the premises and conclusions are combinations. Now R_1-R_{38} have the property that if $\boldsymbol{A}$ and $\boldsymbol{C}$ are combinations, and $\boldsymbol{A}$ conv $\boldsymbol{C}$, then $\boldsymbol{C}$ is derivable from $\boldsymbol{A}$ by R_1-R_{38}. Hence $\bar{\boldsymbol{D}}$ is derivable from $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ by R_1-R_{38}, R_{ik}, R_{ijk}, *i. e.* by R_1-R_{614}.

We now define expressions $\mathfrak{R}_t$ corresponding to the rules R_t $(t = 1, \cdots, 614)$.

For typical rules of the set R_1-R_{38}, the definition of $\mathfrak{R}_t$ follows ($\mathfrak{r}_t$ standing for an expression satisfying the condition $\mathfrak{r}_t(1)$ conv I and the condition given below):

R_1. *If $I(\boldsymbol{p})$, then $\boldsymbol{p}$.*

$$\mathfrak{r}_1(2) \text{ conv } \lambda a \cdot a_2, \qquad \mathfrak{R}_1 \rightarrow \lambda a \cdot \mathfrak{r}_1(\epsilon_1{}^{|a|} \circ \Delta^{a_1}_{[1]}, a).$$

R_2. *If $\boldsymbol{p}$, then $I(\boldsymbol{p})$.*

$$\mathfrak{R}_2 \rightarrow \lambda a \cdot \langle [1], a \rangle.$$

R_3. *If $\boldsymbol{f}(I(\boldsymbol{p}, \boldsymbol{q}))$, then $\boldsymbol{f}(\boldsymbol{p}(\boldsymbol{q}))$.*

$$\mathfrak{r}_3(2) \text{ conv } \lambda a \cdot \langle a_1, \langle a_{212}, a_{22} \rangle \rangle, \qquad \mathfrak{R}_3 \rightarrow \lambda a \cdot \mathfrak{r}_3(\epsilon_3{}^{|a|} \circ \Delta^{a_{211}}_{[1]}, a).$$

R_6. *If $\boldsymbol{f}(\boldsymbol{p}(\boldsymbol{q}, \boldsymbol{p}(\boldsymbol{s}, \boldsymbol{r})))$, then $\boldsymbol{f}(J(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}))$.*

$$\mathfrak{r}_6(2) \text{ conv } \lambda a \cdot \langle a_1, \langle\langle\langle\langle [2], a_{211} \rangle, a_{212} \rangle, a_{222} \rangle, a_{2212} \rangle\rangle,$$
$$\mathfrak{R}_6 \rightarrow \lambda a \cdot \mathfrak{r}_6(\epsilon_4{}^{|a|} \circ \Delta^{a_{2211}}_{a_{2211}}, a).^*$$

If R_t is the rule R_{ik} (for a certain i and k), then $\mathfrak{R}_t$ shall be the expres-

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* The considerations governing the choice of the $\mathfrak{R}_t$ will appear in the proofs of 19XI(t) and 19.23(t). a_2, a_{211}, $\cdots$ are our abbreviations for $\mathfrak{M}_2(a)$, $\mathfrak{M}_1(\mathfrak{M}_1(\mathfrak{M}_2(a)))$, $\cdots$.

sion $\mathfrak{R}_{ik}$ defined thus: Let $\mathfrak{s}_{ik}$ and $\mathfrak{t}_i$ be metads which represent the combinations $\mathfrak{S}_{ik}$ and $\mathfrak{T}_i$, respectively (cf. 19IIIa), and let $\mathfrak{r}_{ik}$ be an expression such that $\mathfrak{r}_{ik}(1)$ conv I and $\mathfrak{r}_{ik}(2)$ conv $\lambda a \cdot \langle \mathfrak{t}_i, a_{12}\rangle$. Let $\mathfrak{R}_{ik} \rightarrow \lambda a \cdot \mathfrak{r}_{ik}(\epsilon_2{}^{|a|} \text{ o } \Delta(a_{11}, \mathfrak{s}_{ik}), a)$.

If R_t is the rule R_{ijk} (for a certain i, j and k), then $\mathfrak{R}_t$ shall be the expression $\mathfrak{R}_{ijk}$ defined thus: Let $\mathfrak{u}_{ij}$ be a metad which represents $\mathfrak{U}_{ij}$, and $\mathfrak{r}_{ijk}$ an expression such that $\mathfrak{r}_{ijk}(1)$ conv $\lambda pq \cdot I^{|p|}(q)$ and $\mathfrak{r}_{ijk}(2)$ conv $\lambda ab \cdot \langle\langle \mathfrak{s}_{jk}, a_2\rangle, b_2\rangle$. Let $\mathfrak{R}_{ijk} \rightarrow \lambda ab \cdot \mathfrak{r}_{ijk}(\epsilon_2{}^{|a|} \text{ o } \epsilon_2{}^{|b|} \text{ o } \Delta(a_{11}, \mathfrak{u}_{ij}) \text{ o } \Delta(b_{11}, \mathfrak{s}_{ik}) \text{ o } \Delta^{a_{12}}_{b_{12}}, a, b)$.

19XI(t). *If the metad* $\boldsymbol{a}$ *represents (the metads* $\boldsymbol{a}$, $\boldsymbol{b}$ *represent) a combination* $\boldsymbol{A}$ *(combinations* $\boldsymbol{A}, \boldsymbol{B}$*) such that* R_t *is applicable to* $\boldsymbol{A}$ *(to the pair* $\boldsymbol{A}, \boldsymbol{B}$*), then* $\mathfrak{R}_t(\boldsymbol{a})(\mathfrak{R}_t(\boldsymbol{a}, \boldsymbol{b}))$ *is a metad which represents the combination resulting from the application.* $(t = 1, \cdots, 614)$.

As illustrative of the arguments for the several values of t, we take the case of a $t > 102$ (≤ 614). Then R_t is the rule R_{ijk}, for a certain i, j and k; and $\boldsymbol{A}$ and $\boldsymbol{B}$ are of the forms $\mathfrak{U}_{ij}(\boldsymbol{f}, \boldsymbol{g})$ and $\mathfrak{S}_{ik}(\boldsymbol{f}, \boldsymbol{p})$, respectively ($\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{p}$ being combinations, by C6II). Then the ranks of $\boldsymbol{A}$ and $\boldsymbol{B}$ are both at least 3. Hence, by 15Ik, 19IIIb and 19I, $\epsilon_2{}^{|\boldsymbol{a}|}$ conv $\epsilon_2{}^{|\boldsymbol{b}|}$ conv 2. $\mathfrak{u}_{ij}$, $\mathfrak{s}_{ik}$ and $\mathfrak{s}_{jk}$ are, by definition, metads which represent the combinations $\mathfrak{U}_{ij}$, $\mathfrak{S}_{ik}$, and $\mathfrak{S}_{jk}$, respectively. Also, by 19IIIf, the metads $\boldsymbol{a}_{11}$, $\boldsymbol{a}_{12}$, $\boldsymbol{a}_2$, $\boldsymbol{b}_{11}$, $\boldsymbol{b}_{12}$, $\boldsymbol{b}_2$ represent the combinations $\mathfrak{U}_{ij}$, $\boldsymbol{f}$, $\boldsymbol{g}$, $\mathfrak{S}_{ik}$, $\boldsymbol{f}$, $\boldsymbol{p}$, respectively. Hence, by 19VI, $\Delta(\boldsymbol{a}_{11}, \mathfrak{u}_{ij})$ conv $\Delta(\boldsymbol{b}_{11}, \mathfrak{s}_{ik})$ conv $\Delta^{\boldsymbol{a}_{12}}_{\boldsymbol{b}_{12}}$ conv 2. Then, by 15Ij, $\epsilon_2{}^{|\boldsymbol{a}|} \text{ o } \epsilon_2{}^{|\boldsymbol{b}|} \text{ o } \Delta(\boldsymbol{a}_{11}, \mathfrak{u}_{ij}) \text{ o } \Delta(\boldsymbol{b}_{11}, \mathfrak{s}_{ik}) \text{ o } \Delta^{\boldsymbol{a}_{12}}_{\boldsymbol{b}_{12}}$ conv 2. Consequently $\mathfrak{R}_{ijk}(\boldsymbol{a}, \boldsymbol{b})$ conv $\mathfrak{r}_{ijk}(2, \boldsymbol{a}, \boldsymbol{b})$, conv $\langle\langle \mathfrak{s}_{jk}, \boldsymbol{a}_2\rangle, \boldsymbol{b}_2\rangle$. By 19V, the latter is a metad which represents $\mathfrak{S}_{jk}(\boldsymbol{g}, \boldsymbol{p})$, which is the formula resulting from the application of R_{ijk} to $\boldsymbol{A}, \boldsymbol{B}$.

COROLLARY. *If the combination* $\bar{\boldsymbol{D}}$ *is derivable from* $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ *by* R_1-R_{614}, *the set of formulas derivable from* $\mathfrak{a}_1, \cdots, \mathfrak{a}_{13}$ *by zero or more operations of passing from* $\boldsymbol{a}$, $\boldsymbol{b}$ *to* $\mathfrak{R}_1(\boldsymbol{a}), \cdots, \mathfrak{R}_{102}(\boldsymbol{a}), \mathfrak{R}_{103}(\boldsymbol{a}, \boldsymbol{b}), \cdots,$ *or* $\mathfrak{R}_{614}(\boldsymbol{a}, \boldsymbol{b})$ *contains a metad which represents* $\bar{\boldsymbol{D}}$.

This follows from the Theorems 19XI(t) by the definition of $\mathfrak{a}_1, \cdots, \mathfrak{a}_{13}$ as metads representing the combinations $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$, respectively.

Now let $\mathfrak{H}$ be an expression which has the properties (1) and (2) of $\boldsymbol{H}$ in 17II when $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_l$, $\boldsymbol{R}_1, \cdots, \boldsymbol{R}_{m+n}$, m, n are taken to be $\mathfrak{a}_1, \cdots, \mathfrak{a}_{13}$, $\mathfrak{R}_1, \cdots, \mathfrak{R}_{614}$, 102, 512, respectively.

19XII. *If the combination* $\bar{\boldsymbol{D}}$ *is representative of a formula* $\boldsymbol{D}$ *provable*

in C_1, then there is a positive integer n such that $\mathfrak{H}(\mathfrak{n})$ *is a metad which represents* $\bar{\boldsymbol{D}}$.

Proof. By 19X, $\bar{\boldsymbol{D}}$ is derivable from $\mathfrak{A}_1, \cdots, \mathfrak{A}_{13}$ by R_1-R_{614}. The conclusion follows by 19XI Cor. and 17II(1) (under our definition of $\mathfrak{H}$).

Let $G \rightarrow \lambda a \cdot \mathfrak{G}(a, \Pi, \Sigma, \&)$.

$$19.22(s) \qquad \mathrm{ad}(\mathfrak{a}_s) G(\mathfrak{a}_s) \qquad (s = 1, \cdots, 13).$$

Proof. Since $\mathfrak{a}_s$ is a given metad, $\mathrm{ad}(\mathfrak{a}_s)$ is provable from the formulas 19.1 by a succession of applications of 19.3. Since $\mathfrak{a}_s$ represents the combination $\mathfrak{A}_s$, which is representative of an axiom $\boldsymbol{A}_s$, $G(\mathfrak{a}_s)$ conv $\mathfrak{G}(\mathfrak{a}_s, \Pi, \Sigma, \&)$, conv $\mathfrak{A}_s(\Pi, \Sigma, \&)$ (19VII), conv $\{\lambda \Pi\Sigma\& \cdot \boldsymbol{A}_s \cdot E(\Pi)\}(\Pi, \Sigma, \&)$, conv $\boldsymbol{A}_s \cdot E(\Pi)$, which is a provable formula.

$$19.23(t): \quad \begin{array}{l} \mathrm{ad}(a)G(a) \supset_a \mathrm{ad}(\mathfrak{R}_t(a))G(\mathfrak{R}_t(a)) \qquad (t = 1, \cdots, 102). \\ {[\mathrm{ad}(a)G(a) \cdot \mathrm{ad}(b)G(b)] \supset_{ab} \mathrm{ad}(\mathfrak{R}_t(a, b))G(\mathfrak{R}_t(a, b))} \\ \qquad (t = 103, \cdots, 614). \end{array}$$

Proof. We take as typical the case of a $t > 102$. Then $\mathfrak{R}_t$ is one of the expressions $\mathfrak{R}_{ijk}$ for a certain i, j and k. 19.22 $\vdash \Sigma ab \cdot \mathrm{ad}(a)G(a) \cdot \mathrm{ad}(b)G(b)$. Assume $\mathrm{ad}(a)G(a) \cdot \mathrm{ad}(b)G(b)$. Since $\mathfrak{u}_{ij}$, $\mathfrak{s}_{ik}$ and $\mathfrak{s}_{jk}$ are given metads, $\mathrm{ad}(\mathfrak{u}_{ij})$, $\mathrm{ad}(\mathfrak{s}_{ik})$ and $\mathrm{ad}(\mathfrak{s}_{jk})$ are provable. Using 14.2, 14.7, 19.2, 19.8, and 19.12, $M(\epsilon_2{}^{|a|} \circ \epsilon_2{}^{|b|} \circ \Delta(a_{11}, \mathfrak{u}_{ij}) \circ \Delta(b_{11}, \mathfrak{s}_{ik}) \circ \Delta^{a_{12}}_{b_{12}})$. Case 1: $\epsilon_2{}^{|a|} \circ \epsilon_2{}^{|b|} \circ \Delta(a_{11}, \mathfrak{u}_{ij}) \circ \Delta(b_{11}, \mathfrak{s}_{ik}) \circ \Delta^{a_{12}}_{b_{12}} = 1$. Then $\mathfrak{R}_{ijk}(a, b) = \mathfrak{r}_{ijk}(1, a, b)$, conv $I^{|a|}(b)$, $= b$ (19.2, 7.2), and $\mathrm{ad}(\mathfrak{R}_{ijk}(a, b))G(\mathfrak{R}_{ijk}(a, b))$ follows from $\mathrm{ad}(b)G(b)$. Case 2: $\epsilon_2{}^{|a|} \circ \epsilon_2{}^{|b|} \circ \Delta(a_{11}, \mathfrak{u}_{ij}) \circ \Delta(b_{11}, \mathfrak{s}_{ik}) \circ \Delta^{a_{12}}_{b_{12}} = 2$. Then $\mathfrak{R}_{ijk}(a, b) = \mathfrak{r}_{ijk}(2, a, b)$, conv $\langle\langle \mathfrak{s}_{jk}, a_2\rangle, b_2\rangle$, and $\mathrm{ad}(\mathfrak{R}_{ijk}(a, b))$ follows from $\mathrm{ad}(\mathfrak{s}_{jk})$, $\mathrm{ad}(a)$ and $\mathrm{ad}(b)$ by means of 19.8 and 19.11. Also $|a| > 2$, $|b| > 2$, $\Delta(a_{11}, \mathfrak{u}_{ij}) = 2$, $\Delta(b_{11}, \mathfrak{s}_{ik}) = 2$, $\Delta^{a_{12}}_{b_{12}} = 2$ (14.2, 14.6, 14.7, 14.9, 19.2, 19.8, 19.12). Now, from $G(a)$ by conversion, $\mathfrak{G}(a, \Pi, \Sigma, \&)$; thence, by 19.18, $\mathfrak{G}(a_1, \mathfrak{G}(a_2), \Pi, \Sigma, \&)$; by another application of 19.18, $\mathfrak{G}(a_{11}, \mathfrak{G}(a_{12}), \mathfrak{G}(a_2), \Pi, \Sigma, \&)$; by two applications of 19.21, $\mathfrak{G}(\mathfrak{u}_{ij}, \mathfrak{G}(b_{12}), \mathfrak{G}(a_2), \Pi, \Sigma, \&)$; and, by conversion (cf. 19VII), $\mathfrak{A}(\Pi, \Sigma, \&)$, where $\mathfrak{A} \rightarrow \mathfrak{U}_{ij}(\mathfrak{G}(b_{12}), \mathfrak{G}(a_2))$. Similarly, from $G(b)$ we infer $\mathfrak{B}(\Pi, \Sigma, \&)$, where $\mathfrak{B} \rightarrow \mathfrak{S}_{ik}(\mathfrak{G}(b_{12}), \mathfrak{G}(b_2))$. If $\mathfrak{C} \rightarrow \mathfrak{S}_{jk}(\mathfrak{G}(a_2), \mathfrak{G}(b_2))$, then $\mathfrak{C}$ is derivable from $\mathfrak{A}$ and $\mathfrak{B}$ by an application of R_{ijk}. Hence, by 19IX, we can infer $\mathfrak{C}(\Pi, \Sigma, \&)$ from $\mathfrak{A}(\Pi, \Sigma, \&)$ and $\mathfrak{B}(\Pi, \Sigma, \&)$. From $\mathfrak{C}(\Pi, \Sigma, \&)$, by conversion, $\mathfrak{G}(\mathfrak{s}_{jk}, \mathfrak{G}(a_2), \mathfrak{G}(b_2), \Pi, \Sigma, \&)$ (19VII); by applications of 19.20,

$\mathfrak{G}(\langle\langle \mathfrak{s}_{jk}, a_2\rangle, b_2\rangle, \Pi, \Sigma, \&)$; by conversion, $G(\langle\langle \mathfrak{s}_{jk}, a_2\rangle, b_2\rangle)$; and thence $G(\mathfrak{R}_{ijk}(a, b))$. Using Axiom 14, $\text{ad}(\mathfrak{R}_{ijk}(a, b)) \cdot G(\mathfrak{R}_{ijk}(a, b))$. By cases (C9I), $\text{ad}(\mathfrak{R}_{ijk}(a, b)) G(\mathfrak{R}_{ijk}(a, b))$.

19.24: $$N(n) \supset_n \text{ad}(\mathfrak{H}(n)) G(\mathfrak{H}(n)).$$

This formula follows from the formulas 19.22(s) and 19.23(t) by 17II(2) and our definition of $\mathfrak{H}$.

19XIII. *If* $\boldsymbol{F}(\boldsymbol{P})$ *is provable in* C_1, *and* $\boldsymbol{P}$ *contains no free symbols, then a formula* $\boldsymbol{U}$ *(containing no free symbols) can be found such that* (1) *if* $\boldsymbol{F}(\boldsymbol{Q})$ *is provable in* C_1, *and* $\boldsymbol{Q}$ *contains no free symbols, then there is a positive integer* q *such that* $\boldsymbol{U}(\boldsymbol{q})$ *conv* $\boldsymbol{Q}$, *and* (2) $N(n) \supset_n \boldsymbol{F}(\boldsymbol{U}(n))$ *is provable.*

Proof. Assume the hypothesis. Let $\boldsymbol{F}'$ and $\boldsymbol{P}'$ be combinations such that $\boldsymbol{F}'$ conv $\lambda p \Pi \Sigma \& \cdot \boldsymbol{F}(p) \cdot E(\Pi)$, and $\boldsymbol{P}'$ conv $\boldsymbol{P}$ (C6V, C5VI, C5V Cor.). Let $\boldsymbol{c}$ be a metad representing $\boldsymbol{F}'(\boldsymbol{P}')$ (19IIIa). Let $\boldsymbol{K}$ be an expression such that $\boldsymbol{K}(1)$ conv $\lambda a \cdot I^{|a|}(\boldsymbol{c})$ and $\boldsymbol{K}(2)$ conv I. Let $\boldsymbol{L} \rightarrow \lambda a \cdot \boldsymbol{K}(\epsilon_1{}^{|a|} \circ \Delta^{a_1}_{c_1}, a)$.

(1) *If the metad* $\boldsymbol{a}$ *represents a combination of the form* $\boldsymbol{F}'(\boldsymbol{Q}')$, *then* $\boldsymbol{L}(\boldsymbol{a})$ *conv* $\boldsymbol{a}$ (15Ij, k, 19I, 19IIIb, f, 19VI).

(2) $\vdash \text{ad}(a) G(a) \supset_a \cdot \text{ad}(\boldsymbol{L}(a)) G(\boldsymbol{L}(a)) \cdot \epsilon(|\boldsymbol{L}(a)|, 1) \circ \Delta(\boldsymbol{L}(a)_1, \boldsymbol{c}_1) = 2.$

Proof. Assume $\text{ad}(a) G(a)$. Case 1: $\epsilon_1{}^{|a|} \circ \Delta^{a_1}_{c_1} = 1$. Then $\boldsymbol{L}(a) = \boldsymbol{K}(1, a)$, $= \boldsymbol{c}$ (19.2, 7.2). 19.1, 19.3 $\vdash \text{ad}(\boldsymbol{c})$; $G(\boldsymbol{c})$ is provable by conversion from $\boldsymbol{F}(\boldsymbol{P}) \cdot E(\Pi)$; and $\epsilon_1{}^{|c|} \circ \Delta^{c_1}_{c_1}$ conv 2. Case 2: $\epsilon_1{}^{|a|} \circ \Delta^{a_1}_{c_1} = 2$. Then $\boldsymbol{L}(a) = \boldsymbol{K}(2, a)$, conv a. In both cases $\text{ad}(\boldsymbol{L}(a)) G(\boldsymbol{L}(a)) \cdot \epsilon(|\boldsymbol{L}(a)|, 1) \circ \Delta(\boldsymbol{L}(a)_1, \boldsymbol{c}_1) = 2$ is provable from the assumptions; and hence, by applications of C9I and Theorem I, (2) holds.

Let $\mathfrak{W} \rightarrow \lambda a \cdot \mathfrak{G}(a_2)$.

(3) *If the metad* $\boldsymbol{a}$ *represents a combination of the form* $\boldsymbol{F}'(\boldsymbol{Q}')$, *then* $\mathfrak{W}(\boldsymbol{a})$ *conv* $\boldsymbol{Q}'$ (19IIIf, 19VII).

(4) $\vdash [\text{ad}(a) G(a) \cdot \epsilon_1{}^{|a|} \circ \Delta^{a_1}_{c_1} = 2] \supset_a F(\mathfrak{W}(a)).$

Proof. By 19.22 and (2), $\Sigma a \cdot \text{ad}(a) G(a) \cdot \epsilon_1{}^{|a|} \circ \Delta^{a_1}_{c_1} = 2$. Assume $\text{ad}(a) G(a) \cdot \epsilon_1{}^{|a|} \circ \Delta^{a_1}_{c_1} = 2$. Then $|a| > 1$ and $\Delta^{a_1}_{c_1} = 2$ (14.6, 14.7, 14.9, 19.2, 19.8, 19.12). Now $G(a)$ conv $\mathfrak{G}(a, \Pi, \Sigma, \&)$; thence, by 19.18, $\mathfrak{G}(a_1, \mathfrak{G}(a_2), \Pi, \Sigma, \&)$; by 19.21, $\mathfrak{G}(\boldsymbol{c}_1, \mathfrak{G}(a_2), \Pi, \Sigma, \&)$. $\mathfrak{G}(\boldsymbol{c}_1, \mathfrak{G}(a_2), \Pi, \Sigma, \&)$

conv $\boldsymbol{F}'(\mathfrak{G}(a_2), \Pi, \Sigma, \&)$ (19IIIf, 19VII), conv $\boldsymbol{F}(\mathfrak{G}(a_2)) \cdot E(\Pi)$ (def. of $\boldsymbol{F}'$), conv $\boldsymbol{F}(\mathfrak{W}(a)) \cdot E(\Pi)$, whence, by Axiom 15, $\boldsymbol{F}(\mathfrak{W}(a))$.

Let $\boldsymbol{U} \rightarrow \lambda n \cdot \mathfrak{W}(\boldsymbol{L}(\mathfrak{H}(n)))$.

(5) Suppose that $\boldsymbol{F}(\boldsymbol{Q})$ is provable in C_1, and that $\boldsymbol{Q}$ contains no free symbols. Let $\boldsymbol{Q}'$ be a combination such that $\boldsymbol{Q}'$ conv $\boldsymbol{Q}$. Then the combination $\boldsymbol{F}'(\boldsymbol{Q}')$ is representative of $\boldsymbol{F}(\boldsymbol{Q})$. Hence, by 19XII, there is a positive integer q such that $\mathfrak{H}(\boldsymbol{q})$ represents $\boldsymbol{F}'(\boldsymbol{Q}')$. Now $\boldsymbol{U}(\boldsymbol{q})$ conv $\mathfrak{W}(\boldsymbol{L}(\mathfrak{H}(\boldsymbol{q})))$, conv $\mathfrak{W}(\mathfrak{H}(\boldsymbol{q}))$ (by (1)), conv $\boldsymbol{Q}'$ (by (3)), conv $\boldsymbol{Q}$.

(6) Assume $N(n)$. By 19.24, ad$(\mathfrak{H}(n)) \cdot G(\mathfrak{H}(n))$. Thence, using (2) and (4), $\boldsymbol{F}(\mathfrak{W}(\boldsymbol{L}(\mathfrak{H}(n))))$, and, by conversion, $\boldsymbol{F}(\boldsymbol{U}(n))$. By Theorem I, $N(n) \supset_n \boldsymbol{F}(\boldsymbol{U}(n))$.

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DOUBLY PERIODIC FUNCTIONS OF THE SECOND KIND AND THE ARITHMETICAL FORM $xy + zw$.

By E. T. Bell.

1. *Introduction.* The sixteen doubly periodic functions of the second kind,

$$\phi_{abc}(x, y) \equiv \vartheta'_1\vartheta_a(x + y)/\vartheta_b(x)\vartheta_c(y),$$

where the triple index abc has the values

001,	010,	023,	032,
100,	111,	122,	133,
203,	212,	221,	230,
302,	313,	320,	331,

give rise to a set of identities of the form

$$\phi_{abc}(x, y)\phi_{rst}(x, -y) \equiv AB + CD, \tag{1}$$

where each of A, B, C, D is a function of x alone, or of y alone, on reducing the numerator $\vartheta'^2_1\vartheta_a(x + y)\vartheta_r(x - y)$ of the left by means of the addition formulas for the thetas. An identity (1) is said to be of the second degree (with reference to the right) in theta quotients provided that each of A, B, C, D has a Fourier expansion in which the coefficients of the several powers of q involve only functions of the divisors of the exponents.

The complete set of identities (1) of the second degree contains a subset of identities from which the entire set can be generated by transformations of the forms

$$\begin{aligned} (2) \qquad & q \to -q, \quad x \to x \pm \pi/2, \quad y \to y \pm \pi/2; \\ & (x, y) \to (y, x), \quad (x, y) \to (x, -y), \quad (x, y) \to (-x, y), \end{aligned}$$

or by repetitions of these, and no identity in the subset is obtainable by these transformations from any other in the subset. It is easily seen that this subset contains precisely 25 identities. By the method of paraphrase, each of these identities implies and is implied by an arithmetical identity concerning parity functions summed over a quadratic partition. The set of 25 arithmetical identities (given in § 4) is thus equivalent to all the identities of the type (1), of the second degree, obtainable from the doubly periodic functions of the second kind, since transformations of the type (2) do not increase or diminish the generality of a parity identity (if the transformations are applied to the trigonometric identity, to which a particular identity (1) is equivalent, before

paraphrasing). In § 5, arithmetical identities of a new, completely general type are indicated.

Let the functions $f(x,y)$, $g(x,y)$ be single-valued and finite for all pairs of integer values of x, y, and beyond the parity conditions

$$(3)\quad f(x,y)=f(-x,-y);\qquad g(x,y)=-g(-x,-y),\quad g(0,0)=0,$$

let f, g be entirely arbitrary. In the notation of parity functions,

$$(4)\qquad f(x,y)\equiv f((x,y)\mid),\qquad g(x,y)\equiv g(\mid(x,y));$$

the parity of f is $(2 \mid 0)$, that of g is $(0 \mid 2)$. In an identity involving functions f or g with integer arguments x, y, all the (x, y) have the same character (x_0, y_0) mod 2, namely, $x\equiv x_0$, $y\equiv y_0$ mod 2. Hence in any such identity we may replace $f(x,y)$, $g(x,y)$ by the functions indicated next, since the transformed functions have the same respective parities as f, g:

$$\begin{aligned}
x\equiv 0 \bmod 2:&\ f(x,y)\to(-1)^{x/2}f(x,y), && g(x,y)\to(-1)^{x/2}g(x,y);\\
y\equiv 0 \bmod 2:&\ f(x,y)\to(-1)^{y/2}f(x,y), && g(x,y)\to(-1)^{y/2}g(x,y);\\
x\equiv 1 \bmod 2:&\ f(x,y)\to(-1\mid x)g(x,y), && g(x,y)\to(-1\mid x)f(x,y);\\
y\equiv 1 \bmod 2:&\ f(x,y)\to(-1\mid y)g(x,y), && g(x,y)\to(-1\mid y)f(x,y);\\
&\ f(x,y)\to f(ax,by), && g(x,y)\to g(ax,by),
\end{aligned}$$

where $(-1\mid x)$ is defined only for odd integers x, and is $(-1)^{(x-1)/2}$, and a, b are arbitrary integers different from zero. These transformations, or others compounded from them, are called the elementary transformations of f, g.

The set of 25 arithmetical identities described above is such that no identity in the set is obtainable from any other by elementary transformations of the functions f, g. The partitions concerned are all of the form $xy+zw$, where x, y, z, w are non-negative integers. With respect to elementary transformations this set of 25 is the irreducible equivalent of the entire set of identities (1) of the second degree. They are obtained from the subset defined in connection with the transformations (2).

2. *Theta identities.* To write out the 25 identities mentioned at the end of § 1 we shall need the following theta quotients.

$$\begin{aligned}
\psi_{10}(x)&\equiv\vartheta_2\vartheta_3\vartheta_1(x)/\vartheta_0(x), & \psi_{02}(x)&\equiv\vartheta_0\vartheta_2\vartheta_0(x)/\vartheta_2(x),\\
\psi_{20}(x)&\equiv\vartheta_0\vartheta_2\vartheta_2(x)/\vartheta_0(x), & \psi_{12}(x)&\equiv\vartheta_0\vartheta_3\vartheta_1(x)/\vartheta_2(x),\\
\psi_{30}(x)&\equiv\vartheta_0\vartheta_3\vartheta_3(x)/\vartheta_0(x), & \psi_{32}(x)&\equiv\vartheta_2\vartheta_3\vartheta_3(x)/\vartheta_2(x);
\end{aligned}$$

$$\begin{aligned}
\psi_{01}(x)&\equiv\vartheta_2\vartheta_3\vartheta_0(x)/\vartheta_1(x), & \psi_{03}(x)&\equiv\vartheta_0\vartheta_3\vartheta_0(x)/\vartheta_3(x),\\
\psi_{21}(x)&\equiv\vartheta_0\vartheta_3\vartheta_2(x)/\vartheta_1(x), & \psi_{13}(x)&\equiv\vartheta_0\vartheta_2\vartheta_1(x)/\vartheta_3(x),\\
\psi_{31}(x)&\equiv\vartheta_0\vartheta_2\vartheta_3(x)/\vartheta_1(x), & \psi_{23}(x)&\equiv\vartheta_2\vartheta_3\vartheta_2(x)/\vartheta_3(x);
\end{aligned}$$

$$\begin{aligned}
\chi_{0123}(x)&\equiv\vartheta_0^2\vartheta_0(x)\vartheta_1(x)/\vartheta_2(x)\vartheta_3(x),\\
\chi_{0213}(x)&\equiv\vartheta_3^2\vartheta_0(x)\vartheta_2(x)/\vartheta_1(x)\vartheta_3(x),
\end{aligned}$$

$$\chi_{0312}(x) \equiv \vartheta_2^2\vartheta_0(x)\vartheta_3(x)/\vartheta_1(x)\vartheta_2(x),$$
$$\chi_{1203}(x) \equiv \vartheta_2^2\vartheta_1(x)\vartheta_2(x)/\vartheta_0(x)\vartheta_3(x),$$
$$\chi_{1302}(x) \equiv \vartheta_3^2\vartheta_1(x)\vartheta_3(x)/\vartheta_0(x)\vartheta_2(x),$$
$$\chi_{2301}(x) \equiv \vartheta_0^2\vartheta_2(x)\vartheta_3(x)/\vartheta_0(x)\vartheta_1(x).$$

The 25 identities of the second degree are as follows:

(I) $\phi_{100}(x, y)\phi_{203}(x, -y) = \psi_{10}(x)\psi_{20}(x) + \psi_{30}(x)\chi_{1203}(y).$

(II) $-\phi_{100}(x, y)\phi_{001}(x, -y) = \psi_{20}(x)\psi_{30}(x) + \psi_{10}(x)\chi_{2301}(y).$

(III) $\phi_{100}(x, y)\phi_{302}(x, -y) = \psi_{10}(x)\psi_{30}(x) + \psi_{20}(x)\psi_{1302}(y).$

(IV) $-\phi_{100}(x, y)\phi_{331}(x, -y) = \psi_{10}(x)\psi_{21}(y) + \psi_{23}(x)\psi_{30}(y).$

(V) $\phi_{100}(x, y)\phi_{032}(x, -y) = \psi_{13}(x)\psi_{30}(y) + \psi_{20}(x)\psi_{12}(y).$

(VI) $\phi_{100}(x, y)\phi_{212}(x, -y) = \psi_{20}(x)\psi_{32}(y) + \psi_{31}(x)\psi_{10}(y).$

(VII) $\phi_{001}(x, y)\phi_{302}(x, -y) = \psi_{10}(x)\psi_{20}(x) + \psi_{30}(x)\chi_{0312}(y).$

(VIII) $-\phi_{001}(x, y)\phi_{331}(x, -y) = \psi_{01}(y)\psi_{31}(y) + \psi_{21}(y)\chi_{1203}(x).$

(IX) $\phi_{001}(x, y)\phi_{320}(x, -y) = \psi_{10}(x)\psi_{20}(y) + \psi_{32}(x)\psi_{31}(y).$

(X) $-\phi_{001}(x, y)\phi_{313}(x, -y) = \psi_{20}(x)\psi_{23}(y) + \psi_{31}(x)\psi_{01}(y).$

(XI) $-\phi_{001}(x, y)\phi_{111}(x, -y) = \psi_{21}(y)\psi_{31}(y) - \psi_{01}(y)\chi_{2301}(x).$

(XII) $\phi_{001}(x, y)\phi_{212}(x, -y) = \psi_{21}(x)\psi_{01}(y) + \psi_{30}(x)\psi_{32}(y).$

(XIII) $\phi_{001}(x, y)\phi_{122}(x, -y) = \psi_{12}(x)\psi_{31}(y) - \psi_{30}(x)\psi_{02}(y).$

(XIV) $-\phi_{001}(x, y)\phi_{221}(x, -y) = \psi_{01}(y)\psi_{21}(y) + \psi_{31}(y)\chi_{1302}(x).$

(XV) $\phi_{111}(x, y)\phi_{212}(x, -y) = \psi_{01}(x)\psi_{31}(x) + \psi_{21}(x)\chi_{0312}(y).$

(XVI) $\phi_{100}(x, y)\phi_{100}(x, -y) = \psi^2_{10}(x) - \psi^2_{10}(y).$

(XVII) $\phi_{100}(x, y)\phi_{133}(x, -y) = \psi_{03}(x)\psi_{30}(y) - \psi_{30}(x)\psi_{03}(y).$

(XVIII) $-\phi_{100}(x, y)\phi_{111}(x, -y) = \psi_{10}(x)\psi_{01}(y) - \psi_{01}(x)\psi_{10}(y).$

(XIX) $\phi_{100}(x, y)\phi_{122}(x, -y) = \psi_{02}(x)\psi_{20}(y) - \psi_{20}(x)\psi_{02}(y).$

(XX) $-\phi_{001}(x, y)\phi_{001}(x, -y) = \psi^2_{20}(x) + \psi^2_{31}(y).$

(XXI) $\phi_{001}(x, y)\phi_{032}(x, -y) = \psi_{03}(x)\psi_{21}(y) + \psi_{31}(x)\psi_{12}(y).$

(XXII) $\phi_{001}(x, y)\phi_{010}(x, -y) = \psi_{01}(x)\psi_{01}(y) - \psi_{10}(x)\psi_{10}(y).$

(XXIII) $\phi_{001}(x, y)\phi_{023}(x, -y) = \psi_{02}(x)\psi_{31}(y) + \psi_{20}(x)\psi_{13}(y).$

(XXIV) $-\phi_{111}(x, y)\phi_{111}(x, -y) = \psi^2_{01}(y) - \psi^2_{01}(x).$

(XXV) $\phi_{111}(x, y)\phi_{122}(x, -y) = \psi_{12}(x)\psi_{21}(y) - \psi_{21}(x)\psi_{12}(y).$

3. *Notation.* The letters m, n, d, δ, t, τ, with or without suffixes or accents, denote integers greater than zero; the n, d, δ, t may be odd or even; the m, τ are always odd. Letters m, n without suffixes denote constants; with suffixes, variables. In referring to previous papers in which parts of this notation were used, it is to be noted that if $n = 2^\alpha m$, $\alpha \geqq 0$, the separation $n = 2^\alpha d\delta$ is identical with the separation $n = t\tau$; namely, either $(2^\alpha d, \delta) = (t, \tau)$ or $(2^\alpha\delta, d) = (t, \tau)$. Similarly for accented letters, or letters with suffixes.

To paraphrase the 25 identities into their arithmetical equivalents we shall need the reduced forms of the Fourier expansions of the theta quotients

on the right of (I)-(XXV). These are given in a previous paper,* together with many more useful in similar work. The series for the ψ are in § 14, p. 172; those for the χ in § 15, p. 173; and those for the ψ^2 in § 16, p. 173, of the paper cited. The only correct list in print for the ϕ is that in § 11 of another paper.† The trigonometric identities in § 8 of that paper are used in reducing products involving sec, csc, tan, ctn to sums of sines or cosines (plus possibly a term in sec, etc.). These expansions and formulas being readily available in the papers cited, we shall not reproduce them here. It will suffice to state only the final results (all of which have been checked), as the method of paraphrase is straightforward and entirely elementary (see the second paper cited). The arithmetical equivalent of a particular identity in § 2 is numbered correspondingly; f, g are as in § 1 (3), and summations refer to all values of the variables (also to the specified divisors of the constants) in the partitions indicated in each instance.

One detail in reading the identities may be noted. In (II), for example, the outer Σ (without limits) on the right refers to all t, τ defined by the given partition, and so in all similar cases. By introducing appropriate functions of divisors, as $\zeta'_0(n)$ in (VIII), for example, reductions of such sums are sometimes possible. However, it is usually simpler to leave the identities without such reductions.

4. *Parity identities.* For the m, n, d, δ, t, τ notation see § 3. The (d, δ) and the (t, τ), with or without suffixes or accents, denote pairs of conjugate divisors, and a particular pair refers to the m or n in the stated partition that has the same display of suffixes and accents as those in the particular pair. For example, if the partition is $n = m_i + n_j$, and the pairs (t_i, τ_i), (d_j, δ_j), (d, δ) occur in the parity identity, (t_i, τ_i) refer to m_i, (d_j, δ_j) to n_j, (d, δ) to n. Thus, written in full, the partition would be $n = m_i + n_j$, $n = d\delta$, $m_i = t_i\tau_i$, $n_j = d_j\delta_j$. This convention saves much space. Notice in particular that if the partition contains numerical factors, as in $an = bm_i + cn_j$, where a, b, c are definite integers, the (d, δ), (t_i, τ_i), (d_j, δ_j) refer to the divisors of n, m_i, n_j, and *not* to those of an, bm_i, cn_j. Note that the pairs of conjugates are (d, δ) and (t, τ); (d, t), (d, τ), (δ, t), (δ, τ) do not occur.

(I_1) $\quad 2m = m_1 + m_2;\ \ m = 2n_3 + m_4:$

$$\Sigma(-1 \mid \tau_2)[g(t_1 + t_2, \tau_1 - \tau_2) + g(t_1 - t_2, \tau_1 + \tau_2) - g(t_1 + t_2, 0) - g(t_1 - t_2, 0)] - 2\Sigma(-1 \mid \tau_3)[g(4t_3, 2t_4) - g(4t_3, -2t_4)] = \Sigma g(0, 2t).$$

* *Messenger of Mathematics*, vol. 54 (1924), pp. 166-176.

† *Transactions of the American Mathematical Society*, vol. 22 (1921), pp. 198-219.

$(\mathrm{I}_2)\qquad 4n = m_1 + m_2,\ 2n = m_3 + m_4$:

$$\Sigma(-1\mid\tau_2)[g(t_1+t_2, \tau_1-\tau_2) + g(t_1-t_2, \tau_1+\tau_2) - g(t_1+t_2, 0) \\ - g(t_1-t_2, 0)] - 2\Sigma(-1\mid\tau_3)[g(2t_3, 2t_4) - g(2t_3, -2t_4)] = 0.$$

$(\mathrm{II})\qquad m = m_1 + 2n_2$:

$$2\Sigma[f(t_1+2t_2, \tau_1-\tau_2) - f(t_1-2t_2, \tau_1+\tau_2) \\ - (-1\mid\tau_1\tau_2)\{f(t_1+2t_2, 0) + f(t_1-2t_2, 0)\} \\ - (-1)^{\delta_2}\{f(t_1, -2d_2) - f(t_1, 2d_2)\}] \\ = \Sigma[\{(-1\mid\tau) - 1\}f(t, 0) - 2\sum_{r=1}^{(\tau-1)/2} f(t, 2r)].$$

In this we have the first instance of one of the variables in the partition, here n_2, being separated into pairs of conjugate divisors of different types, namely, $n_2 = t_2\tau_2$ and $n_2 = d_2\delta_2$. The second type can be reduced to the first, but the above statement is the simpler. Similarly in several subsequent identities.

$(\mathrm{III})\qquad m = m_1 + 2n_2$:

$$2\Sigma(-1\mid\tau_2)[g(t_1+2t_2, \tau_1-\tau_2) + g(t_1-2t_2, \tau_1+\tau_2) - g(t_1+2t_2, 0) \\ - g(t_1-2t_2, 0)] + 2\Sigma(-1\mid\tau_1)(-1)^{n_2+d_2+\delta_2}[g(t_1, 2d_2) - g(t_1, -2d_2)] \\ = \Sigma[\{1 - (-1\tau)\}g(t, 0) - 2(-1\mid\tau)\sum_{r=1}^{(\tau-1)/2}(-1)^r g(t, 2r)].$$

$(\mathrm{IV})\qquad m = m_1 + 2n_2 = m_3 + 4n_4$:

$$2\Sigma[(-1)^{n_2}f(t_1+2t_2, \tau_1-\tau_2) - f(t_1-2t_2, \tau_1+\tau_2) \\ - (-1\mid t_1\tau_2)\{f(t_1, 2t_2) + f(t_1, -2t_2)\}] \\ + 2\Sigma(-1)^{\delta_4}[f(t_3, 2d_4) - f(t_3, -2d_4)] \\ = \Sigma[\{(-1\mid t) - 1\}f(t, 0) - 2\sum_{r=1}^{(\tau-1)/2} f(t, 2r)].$$

$(\mathrm{V})\qquad m = m_1 + 2n_2 = m_3 + 4n_4$:

$$2\Sigma(-1\mid\tau_2)[(-1)^{n_2}\{g(t_1+2t_2, \tau_1-\tau_2) + g(t_1-2t_2, \tau_1+\tau_2)\} \\ - (-1\mid m_1)\{g(t_1, 2t_2) + g(t_1, -2t_2)\}] \\ + 2\Sigma(-1\mid\tau_3)(-1)^{d_4+\delta_4}[g(t_3, 2d_4) - g(t_3, -2d_4)] \\ = \Sigma[\{(-1\mid m) - (-1\mid\tau)\}g(t, 0) - 2(-1\mid\tau)\sum_{r=1}^{(\tau-1)/2}(-1)^r g(t, 2r)].$$

$(\mathrm{VI})\qquad m = m_1 + 2n_2 = m_3 + 4n_4$:

$$\Sigma[(-1\mid\tau_1\tau_2)\{f(t_1, \tau_2) + f(t_1, -\tau_2)\} + (-1)^{n_2}\{f(\tau_2, -t_1) - f(\tau_2, t_1)\}] \\ + \Sigma\ (-1)^{\delta_4}[f(t_3+2d_4, \tau_3-2\delta_4) - f(t_3-2d_4, \tau_3+2\delta_4)] \\ = \Sigma\sum_{r=1}^{(\tau-1)/2}[f(2r-1, t) + (-1\mid\tau)(-1)^r f(t, 2r-1)].$$

$(\mathrm{VII})\qquad n = n_1 + n_2 = n_3 + 2n_4;\ 2n = m_5 + m_6$:

$$\Sigma(-1\mid\tau_2)[g(2t_1+2t_2, \tau_1-\tau_2) + g(2t_1-2t_2, \tau_1+\tau_2)] \\ - 2\Sigma(-1\mid\tau_3)[g(2t_3, 2\tau_4) - g(2t_3, -2\tau_4)]$$

$$-\Sigma(-1\mid\tau_6)[g(t_5+t_6,0)+g(t_5-t_6,0)]$$
$$=\tfrac{1}{2}\{1+(-1)^n\}\Sigma g(0,2\tau)-\Sigma(-1\mid\tau)[g(2t,0)$$
$$+\sum_{r=1}^{(\tau-1)/2}\{(-1)^r g(2t,\,2r)+g(2t,-2r)\}].$$

(VIII) $n=n_1+n_2=m_3+2n_4$:

$$2\Sigma(-1)^{n_2}[f(t_1+t_2,\tau_1-\tau_2)-f(t_1-t_2,\tau_1+\tau_2)+f(0,\tau_1+\tau_2)-f(0,\tau_1-\tau_2)]$$
$$+4\Sigma(-1)^{\delta_4}[f(\tau_3,2d_4)-f(\tau_3,-2d_4)]$$
$$=\{1+(-1)^n\}[\zeta'_0(n)f(0,0)-\Sigma f(t,0)]$$
$$-2\Sigma\sum_{r=1}^{(\tau-1)/2}[f(t,2r)+(-1)^n f(t,-2r)-\{1+(-1)^n\}f(0,2r)].$$

Here $\zeta'_0(n)\equiv$ the number of odd divisors of n.

(IX) $n=n_1+n_2$; $2n=m_3+m_4$:

$$\Sigma[(-1\mid\tau_2)g(2t_1+\tau_2,\tau_1-2t_2)+g(2t_1-\tau_2,\tau_1+2t_2)$$
$$-(-1\mid\tau_1)(-1)^{n_2}\{g(\tau_1,\tau_2)-g(\tau_1,-\tau_2)\}]$$
$$-\Sigma(-1\mid\tau_4)[g(t_3,t_4)+g(t_3,-t_4)]$$
$$=\Sigma\sum_{r=1}^{t}[(-1\mid\tau)g(-\tau,2r-1)-(-1)^{t+r}g(2r-1,\tau)].$$

(X) $n=n_1+n_2$; $2n=m_3+m_4$:

$$\Sigma[(-1)^{n_2}\{f(2t_1-\tau_2,\tau_1+2t_2)-f(2t_1+\tau_2,\tau_1-2t_2)$$
$$-(-1)^{n_1}f(\tau_1,-\tau_2)-f(\tau_1,\tau_2)]-\Sigma(-1\mid\tau_3\tau_4)[f(t_3,\tau_4)+f(t_3,-\tau_4)]$$
$$=\Sigma\sum_{r=1}^{t}[(-1)^n f(\tau,-2r+1)-f(2r-1,\tau)].$$

(XI$_1$) $m=m_1+2n_2=n_3+n_4$:

$$\Sigma[f(t_1+d_2,\tau_1-2\delta_2)-f(t_1-d_2,\tau_1+2\delta_2)+(-1)^{\delta_2}f(0,\tau_1-2d_2)$$
$$-f(0,\tau_1+2d_2)]-\Sigma(-1)^{\delta_3}[f(d_3,\tau_4)-f(d_3,-\tau_4)]$$
$$=\Sigma[\sum_{r=1}^{t-1}f(r,\tau)-\sum_{r=1}^{(\tau-1)/2}\{f(t,2r-1)+f(0,2r-1)\}].$$

(XI$_2$) $2n=n_1+n_2$; $n=n_3+n_4$:

$$\Sigma[(-1)^{\delta_4}\{f(0,\tau_3+2d_4)-f(0,\tau_3-2d_4)+f(2t_3+d_4,\tau_3-2\delta_4)$$
$$-f(2t_3-d_4,\tau_3+2\delta_4)\}]-\Sigma(-1)^{\delta_1}[f(d_1,\tau_2)-f(d_1,-\tau_2)]$$
$$=\Sigma[\sum_{r=1}^{\delta}\{(-1)^d f(0,2r-1)-f(d,-2r+1)\}+f(0,\tau)+\sum_{r=1}^{2t-1}f(r,\tau)$$
$$+\sum_{r=1}^{(\tau-1)/2}\{f(0,2r-1)-f(2t,2r-1)\}].$$

(XII$_1$) $m=m_1+2n_2=n_3+n_4$:

$$\Sigma(-1)^{\delta_2}[f(t_1-d_2,\tau_1+2\delta_2)-f(t_1+d_2,\tau_1-2\delta_2)+f(d_2,\tau_1)-f(d_2,-\tau_1)]$$
$$-\Sigma(-1\mid\tau_3\tau_4)[f(t_3,\tau_4)+f(t_3,-\tau_4)]$$
$$=\Sigma[\tfrac{1}{2}\{(-1\mid\tau)-1\}f(0,\tau)-\sum_{r=1}^{\tau-1}f(r,t)-(-1\mid\tau)\sum_{r=1}^{(\tau-1)/2}(-1)^r f(t,2r-1)].$$

(XII$_2$) $n = n_1 + n_2;\ 2n = n_3 + n_4$:

$$\Sigma(-1)^{\delta_2}[f(2t_1 - d_2, \tau_1 + 2\delta_2) - f(2t_1 + d_2, \tau_1 - 2\delta_2) + f(d_2, \tau_1) - f(d_2, -\tau_1)]$$
$$-\Sigma(-1 \mid \tau_3\tau_4)\,[f(t_3, \tau_4) + f(t_3, -\tau_4)]$$
$$= \Sigma[(-1)^{\delta}\sum_{r=1}^{\delta} f(d, -2r+1) + \tfrac{1}{2}\{(-1 \mid \tau) - 1\}f(0, \tau) - \sum_{r=1}^{2t-1} f(r, \tau)$$
$$-(-1 \mid \tau)\sum_{r=1}^{(\tau-1)/2}(-1)^r f(2t, 2r-1)].$$

(XIII$_1$) $m = m_1 + 2n_2 = n_3 + n_4$:

$$2\Sigma(-1)^{d_2+\delta_2}[f(t_1 + d_2, \tau_1 - 2\delta_2) - f(t_1 - d_2, \tau_1 + 2\delta_2) + f(d_2, \tau_1) - f(d_2, -\tau_1)]$$
$$+2\Sigma(-1 \mid \tau_3\tau_4)(-1)^{n_4}[f(t_3, \tau_4) + f(t_3, -\tau_4)]$$
$$= 2\Sigma[(-1 \mid \tau)\sum_{r=1}^{(\tau-1)/2}(-1)^r f(t, 2r-1) - \sum_{r=1}^{\tau-1}(-1)^r f(r, t)]$$
$$+\Sigma[(-1 \mid \tau) - 1]f(0, \tau).$$

(XIII$_2$) $n = n_1 + n_2,\ 2n = n_3 + n_4$:

$$2\Sigma(-1)^{d_2+\delta_2}[f(2t_1 + d_2, \tau_1 - 2\delta_2) - f(2t_1 - d_2, \tau_1 + 2\delta_2) - f(d_2, \tau_1) + f(d_2, -\tau_1)]$$
$$+2\Sigma(-1 \mid \tau_3\tau_4)(-1)^{n_4}[f(t_3, \tau_4) + f(t_3, -\tau_4)]$$
$$= 2\Sigma[\sum_{r=1}^{2t-1}(-1)^r f(r, \tau) + (-1 \mid \tau)\sum_{r=1}^{(\tau-1)/2}(-1)^r f(2t, 2r-1)]$$
$$+\Sigma[1 - (-1 \mid \tau)]f(0, \tau) - 2\Sigma(-1)^{d+\delta}\sum_{r=1}^{\delta} f(d, -2r+1).$$

(XIV$_1$) $m = n_1 + n_2 = m_3 + 2n_4$:

$$\Sigma(-1)^{d_2+\delta_2}[f(d_2, \tau_1) - f(d_2, -\tau_1)]$$
$$+\Sigma(-1)^{\delta_4}[f(t_3 + \delta_4, \tau_3 - 2d_4) - f(t_3 - \delta_4, \tau_3 + 2d_4)$$
$$+f(0, \tau_3 + 2d_4) - f(0, \tau_3 - 2d_4)]$$
$$= \Sigma[\sum_{r=1}^{(\tau-1)/2}\{f(0, 2r-1) - f(t, 2r-1)\} - \sum_{r=1}^{\tau-1}(-1)^r f(r, t)].$$

(XIV$_2$) $2n = n_1 + n_2,\ n = n_3 + n_4$:

$$\Sigma(-1)^{d_2+\delta_2}[f(d_2, -\tau_1) - f(\mathrm{d}_2, \tau_1)]$$
$$+\Sigma(-1)^{\delta_4}[f(0, \tau_3 + 2d_4) - f(0, \tau_3 - 2d_4)$$
$$+f(2t_3 + \delta_4, \tau_3 - 2d_4) - f(2t_3 - \delta_4, \tau_3 + 2d_4)]$$
$$= \Sigma[f(0, \tau) + \sum_{r=1}^{(\tau-1)/2}\{f(0, 2r-1) - f(2t, 2r-1)\} + \sum_{r=1}^{2t-1}(-1)^r f(r, \tau)]$$
$$+\Sigma(-1)^d\sum_{r=1}^{\delta}[f(0, 2r-1) - f(d, -2r+1)].$$

(XV) $n = n_1 + n_2,\ 2n = n_3 + n_4$:

$$2\Sigma(-1)^{\delta_2}[f(2d_1 - 2d_2, \delta_1 + \delta_2) - f(2d_1 + 2d_2, \delta_1 - \delta_2)$$
$$-2f(2d_2, -\tau_1) + 2f(2d_2, \tau_1)] - 2\Sigma(-1)^{n_4}[f(\tau_3 - \tau_4, 0) - f(\tau_3 + \tau_4, 0)]$$
$$= 2\Sigma[f(0, 0) + 2\sum_{r=1}^{(\tau-1)/2} f(2r, 0)] + \Sigma[2(-1)^{\delta}f(2d, 0) - \{1 + (-1)^{\delta}\}f(0, \delta)$$
$$-2\sum_{r=1}^{\delta-1}\{f(2r, d) + (-1)^d f(2r, -d) - (-1)^{\delta}f(2d, -r) - (-1)^{\delta+r}f(2d, r)\}].$$

(XVI) $2n = m_1 + m_2$:

$$\Sigma[f(t_1 - t_2, \tau_1 + \tau_2) - f(t_1 + t_2, \tau_1 - \tau_2)] = \Sigma t[f(0, 2t) - f(2t, 0)].$$

(XVII$_1$) $2m = m_1 + m_2,\ m = n_3 + n_4$:

$$\Sigma(-1 \mid m_2)[f(t_1 - t_2, \tau_1 + \tau_2) - f(t_1 + t_2, \tau_1 - \tau_2)]$$
$$+ \Sigma(-1)^{n_3}(-1 \mid \tau_3\tau_4)[f(2t_4, 2t_3) - f(2t_3, 2t_4) + f(2t_4, -2t_3) - f(2t_3, -2t_4)]$$
$$= \Sigma(-1 \mid \tau)[f(0, 2t) - f(2t, 0)].$$

(XVII$_2$) $4n = m_1 + m_2,\ 2n = n_3 + n_4$:

$$\Sigma(-1 \mid m_2)[f(t_1 - t_2, \tau_1 + \tau_2) - f(t_1 + t_2, \tau_1 - \tau_2)]$$
$$= \Sigma(-1)^{n_3}(-1 \mid \tau_3\tau_4)[f(2t_3, 2t_4) - f(2t_4, 2t_3) + f(2t_3, -2t_4) - f(2t_4, -2t_3)].$$

(XVIII) $m = m_1 + 4n_2 = m_3 + 2n_4$:

$$\Sigma[f(t_1 - 2d_2, \tau_1 + 2\delta_2) - f(t_1 + 2d_2, \tau_1 - 2\delta_2)]$$
$$+ \Sigma[f(t_3, -\tau_4) - f(t_3, \tau_4) + f(\tau_4, t_3) - f(\tau_4, -t_3)]$$
$$= \sum_{r=1}^{(\tau-1)/2}[f(t, 2r - 1) - f(2r - 1, t)].$$

(XIX) $m = m_1 + 4n_2 = m_3 + 2n_4$:

$$\Sigma(-1)^{d_2+\delta_2}[f(t_1 + 2d_2, \tau_1 - 2\delta_2) - f(t_1 - 2d_2, \tau_1 + 2\delta_2)]$$
$$+ \Sigma(-1 \mid \tau_3\tau_4)(-1)^{n_4}[f(t_3, \tau_4) + f(t_3, -\tau_4) - f(\tau_4, t_3) - f(\tau_4, -t_3)]$$
$$= \Sigma(-1 \mid \tau)\sum_{r=1}^{(\tau-1)/2}(-1)^r[f(t, 2r - 1) - f(2r - 1, t)].$$

(XX$_1$) $m = n_1 + n_2$:

$$2\Sigma[f(t_1 + t_2, \tau_1 - \tau_2) - f(t_1 - t_2, \tau_1 + \tau_2)]$$
$$= \Sigma\left[(t - 1)f(t, 0) - \sum_{r=1}^{(\tau-1)/2}\{f(t, 2r) + f(t, -2r)\}\right].$$

(XX$_2$) $2n = n_1 + n_2$:

$$2\Sigma[f(t_1 + t_2, \tau_1 - \tau_2) - f(t_1 - t_2, \tau_1 + \tau_2)]$$
$$= \Sigma\left[(2t - 1)f(2t, 0) - \sum_{r=1}^{(\tau-1)/2}\{f(2t, 2r) + f(2t, -2r)\} - df(0, 2d)\right]$$

(XXI$_1$) $m = n_1 + n_2 = m_3 + 2n_4$:

$$\Sigma(-1)^{n_2}(-1 \mid \tau_2)[g(t_1 + t_2, \tau_1 - \tau_2) + g(t_1 - t_2, \tau_1 + \tau_2)]$$
$$+ \Sigma(-1)^{\delta_4}(-1 \mid \tau_3)\{1 + (-1)^{d_4}\}[g(t_3, 2d_4) - g(t_3, -2d_4)]$$
$$= \Sigma(-1 \mid \tau)\sum_{r=1}^{(\tau-1)/2}[g(t, -2r) - (-1)^r g(t, 2r)].$$

(XXI$_2$) $2n = n_1 + n_2,\ n = n_3 + n_4$:

$$2\Sigma(-1)^{n_2}(-1 \mid \tau_2)[g(t_1 + t_2, \tau_1 - \tau_2) + g(t_1 - t_2, \tau_1 + \tau_2)]$$
$$- 2\Sigma(-1)^{\delta_4}(-1 \mid \tau_3)\{1 - (-1)^{d_4}\}[g(2t_3, 2d_4) - g(2t_3, -2d_4)]$$
$$= -2\Sigma(-1 \mid \tau)\left[g(2t, 0) + \sum_{r=1}^{(\tau-1)/2}\{(-1)^r g(2t, 2r) + g(2t, -2r)\}\right]$$
$$+ \Sigma(-1)^{\delta}\{1 - (-1)^{d}\}g(0, 2d).$$

(XXII) $2n = n_1 + n_2 = m_3 + m_4$:

$$\Sigma[f(2t_1 + \tau_2, \tau_1 - 2t_2) - f(2t_1 - \tau_2, \tau_1 + 2t_2) + f(\tau_1, -\tau_2) - f(\tau_1, \tau_2)]$$
$$+ \Sigma[f(t_3, t_4) - f(t_3, -t_4)]$$
$$= \Sigma[\sum_{r=1}^{t} \{f(2r - 1, \tau) - f(\tau, -2r + 1)\}].$$

(XXIII) $n = n_1 + n_2,\ 2n = m_3 + m_4$:

$$\Sigma(-1 \mid \tau_1)[(-1)^{n_1}\{g(2t_2 + \tau_1, \tau_2 - 2t_1) + g(2t_2 - \tau_1, \tau_2 + 2t_1)$$
$$- (-1)^{n}\{g(\tau_1, \tau_2) - g(\tau_1, -\tau_2)\}]$$
$$- \Sigma(-1 \mid \tau_3 m_4)[g(t_3, t_4) - g(t_3, -t_4)]$$
$$= - (-1)^{n} \sum_{r=1}^{t} [(-1 \mid \tau) g(\tau, -2r + 1) + (-1)^{r} g(2r - 1, \tau)].$$

(XXIV) $n = n_1 + n_2$:

$$\Sigma[f(d_1 - d_2, \delta_1 + \delta_2) - f(d_1 + d_2, \delta_1 - \delta_2)]$$
$$= \Sigma[(d - 1)\{f(0, d) - f(d, 0)\}$$
$$+ \sum_{r=1}^{\delta-1} \{f(d, r) - f(r, d) + f(d, -r) - f(r, -d)\}].$$

(XXV) $n = n_1 + n_2$:

$$2\Sigma(-1)^{d_2+\delta_2}[f(d_1 + d_2, \delta_1 - \delta_2) - f(d_1 - d_2, \delta_1 + \delta_2)$$
$$+ (-1)^{\delta_1}\{f(d_1, d_2) - f(d_1, -d_2) - f(d_2, d_1) + f(d_2, -d_1)\}]$$
$$= \Sigma(-1)^{\delta}[\{1 + (-1)^{d}\}\{f(0, d) - f(d, 0)\}$$
$$+ 2 \sum_{r=1}^{\delta-1} \{(-1)^{d} f(r, -d) - (-1)^{d} f(d, -r)$$
$$+ (-1)^{r} f(r, d) - (-1)^{r} f(d, r)\}].$$

From this set many more can be written down, by elimination of a particular partition, etc., but the set as given is probably in the simplest form.

5. *General Identities.* We shall not take space here to write these out, but will reserve them for another occasion. It will be noticed that the arguments of f or of g in several pairs of the identities in § 4 are the same, and that one identity in particular pairs of this kind involves f, the other, g. Hence each such pair is equivalent to a single identity involving the function $h(x, y)$, which is finite and single-valued when x, y are simultaneously integers, and which otherwise is *completely arbitrary.* For, we may write

$$h(x, y) \equiv \tfrac{1}{2}[h(x, y) + h(-x, -y)] + \tfrac{1}{2}[h(x, y) - h(-x, -y)],$$

and the first [ ] is an *instance* of $f(x, y)$, the second, of $g(x, y)$. These identities involving h are applicable to certain arithmetical forms of arbitrary degree.

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DETERMINATION OF THE GROUPS OF ORDERS 162-215 OMITTING ORDER 192.

By J. K. SENIOR and A. C. LUNN.

The groups of order g where $100 < g < 162$ and $g \neq 128$ have recently been listed,* and it is a comparatively easy matter to treat the cases where $161 < g < 216$ and $g \neq 192$. The present paper is therefore a continuation of the one just cited, and the methods and symbolism used are the same as those therein defined.

Between 161 and 216 there are only 7 integers which are the product of more than four prime factors. These are

$162 = 2 \cdot 3^4$	$180 = 2^2 \cdot 3^2 \cdot 5$	$208 = 2^4 \cdot 13$
$168 = 2^3 \cdot 3 \cdot 7$	$192 = 2^6 \cdot 3$	
$176 = 2^4 \cdot 11$	$200 = 2^3 \cdot 5^2$	

The groups of order 168 have been listed by G. A. Miller †; those of orders 176 and 208 by Lunn and Senior.‡ To determine the number groups of order 192 is very laborious, and no attempt is made here to solve the problem. But brief arguments suffice to cover the orders 162, 180 and 200 which are here treated in some detail. For the orders where g is the product of less than five factors, since the general methods are known, only the results are given.

THE GROUPS OF ORDER $162 = 2 \cdot 3^4$.

Every group of order 162 is solvable and thus determines a $(G^{81}_{81k} : G^2{}_2)_k$, $(k = 1$ or $2)$. Hence every group of order 162 occurs in one of the following divisions:

Division (a) $(G^{81}_{81} : G^2{}_2)_1$ Division (b) $(G^{81}_{162} : G^2{}_2)_2$.

Division (a). $(G^{81}_{81} : G^2{}_2)_1$. A group in this division is the direct product of its Sylow subgroups. Since there are fifteen groups of order 81, and one group of order 2, there are fifteen groups of division (a). Five of these are abelian.

Division (b). $(G^{81}_{162} : G^2{}_2)_2$. A group in this division corresponds to a set of conjugate subgroups of order 2 in the i-group of a group of order 81. The fifteen groups of this latter order are therefore considered one at a time. In the case of each i-group, the number of sets of conjugate subgroups of order 2 has been proven by the authors, but, in order not to expand the treatment unduly, the proofs are here omitted and only the results given. In the following table, each group of order 81 or 162 is defined by the relations of its generators, which are labelled A–E.

* Senior and Lunn, *American Journal of Mathematics*, vol. 56 (1934), p. 328.
† G. A. Miller, *American Mathematical Monthly*, vol. 9 (1902), p. 1.
‡ Lunn and Senior, *American Journal of Mathematics*, vol. 56 (1934), p. 321.

TABLE I.

	Groups of Order 81	$B^{-1}AB$	$C^{-1}AC$	$C^{-1}BC$	$D^{-1}AD$	$D^{-1}BD$	$D^{-1}CD$	Groups of Order 162	$E^{-1}AE$	$E^{-1}BE$	$E^{-1}CE$	$E^{-1}DE$	$E^2=1$
1	$A^{81}=1$							1	A^{-1}				
2	$A^{27}=B^3=1$	A						2	A^{-1}	B			
								3	A	B^{-1}			
								4	A^{-1}	B^{-1}			
3	$A^{27}=B^3=1$	A^{10}						5	A^{-1}	B			
4	$A^9=B^9=1$	A						6	A	B^{-1}			
								7	A^{-1}	B^{-1}			
5	$A^9=B^3=C^3=1$	A	A	B				8	A	B	C^{-1}		
								9	A	B^{-1}	C^{-1}		
								10	A^{-1}	B	C		
								11	A^{-1}	B	C^{-1}		
								12	A^{-1}	B^{-1}	C^{-1}		
6	$A^9=B^3=1 \quad C^3=A^3$	A	A^4B	BA^6				13	A^{-1}	B	C^{-1}		
7	$A^9=B^3=C^3=1$	A	A^4B	BA^6				14	A^{-1}	B^{-1}	C		
								15	A	B^{-1}	C^{-1}		
								16	A^{-1}	B	C^{-1}		
8	$A^9=B^3=C^3=1$	A	A^7B	BA^3				17	A^{-1}	B^{-1}	C		
								18	A	B^{-1}	C^{-1}		
								19	A^{-1}	B	C^{-1}		
9	$A^9=B^3=C^3=1$	A	A^4	B				20	A	B^{-1}	C		
								21	A^{-1}	B	C		
								22	A^{-1}	B^{-1}	C		
10	$A^9=B^3=C^3=1$	A	A	BA^3				23	A	B^{-1}	C^{-1}		
								24	A^{-1}	B^{-1}	C		
11	$A^9=B^9=1$	A^4						25	A^{-1}	B			
12	$A^9=B^3=C^3=1$	A	AB	B				26	A	B^{-1}	C^{-1}		
								27	A^{-1}	B	C^{-1}		
								28	A^{-1}	B^{-1}	C		
13	$A^3=B^3=C^3=D^3=1$	A	A	B	A	B	C	29	A	B	C	D^{-1}	
								30	A	B	C^{-1}	D^{-1}	
								31	A	B^{-1}	C^{-1}	D^{-1}	
								32	A^{-1}	B^{-1}	C^{-1}	D^{-1}	
14	$A^3=B^3=C^3=D^3=1$	A	A	BA	A	B	C	33	A	B	C	D^{-1}	
								34	A	B^{-1}	C^{-1}	D	
								35	A^{-1}	B	C^{-1}	D	
								36	A	B^{-1}	C^{-1}	D^{-1}	
								37	A^{-1}	B	C^{-1}	D^{-1}	
15	$A^3=B^3=C^3=D^3=1$	A	A	B	A	BA	CBA^{-1}	38	A^{-1}	B^{-1}	C^{-1}	D	
								39	A	B^{-1}	C	D^{-1}	
								40	A^{-1}	B	C^{-1}	D^{-1}	

The number of groups of order 162 is thus:

Division (a)	Division (b)	Total
15	40	55

THE GROUPS OF ORDER $180 = 2^2 \cdot 3^2 \cdot 5$.

It is well known that there is just one insolvable group of order 180. A solvable group of this order determines a $(G^4{}_{4k_1} : G^9{}_{9k_2} : G^5{}_{5k_3})_{uv}$ $(uv = k_1k_2k_3)$.

$k_1 = 1$ or 3, since 4 and 12 are the only orders which divide 180 for which transitive groups of degree four exist.

$k_2 = 1$, 2, or 4, since 9, 18 and 36 are the only orders which divide 180 for which transitive groups of degree nine exist.

$k_3 = 1$, 2, or 4, since 5, 10 and 20 are the only orders for which solvable transitive groups of degree five exist.

Thus every solvable group of order 180 occurs in one of the following divisions:

Division (a)	$(G^4{}_4 : G^9{}_9 : G^5{}_5)_1$	Division (f)	$[(G^5{}_{20} : G^4{}_4)_4 : G^9{}_{18}]_2$
" (b)	$[(G^4{}_4 : G^9{}_{18})_2 : G^5{}_5]_1$	" (g)	$[(G^5{}_{20} : G^4{}_4)_4 : G^9{}_{36}]_4$
" (c)	$[(G^4{}_4 : G^5{}_{10})_2 : G^9{}_9]_1$	" (h)	$[(G^9{}_{36} : G^4{}_4)_4 : G^5{}_5]_1$
" (d')	$[(G^9{}_{18} : G^5{}_{10})_2 : G^4{}_4]_2$	" (i)	$[(G^9{}_{36} : G^4{}_4)_4 : G^5{}_{10}]_2$
" (d'')	$[(G^9{}_{18} : G^5{}_{10})_1 : G^4{}_4]_4$	" (j)	$[(G^4{}_{12} : G^9{}_{9k_2})_{3k_2} : G^5{}_{5k_3}]_{k_3}$
" (e)	$[(G^5{}_{20} : G^4{}_4)_4 : G^9{}_9]_1$		

Division (a). $(G^4{}_4 : G^9{}_9 : G^5{}_5)_1$. A group in this division is the direct product of its Sylow subgroups. There are two groups of order four, two of order nine, and one of order five. Hence there are $2 \times 2 = 4$ groups of division (a). They are all abelian.

Division (b). $[(G^4{}_4 : G^9{}_{18})_2 : G^5{}_5]_1$. Each of the two groups $G^4{}_4$ can be dimidiated in one way with each of the three groups $G^9{}_{18}$. Hence there are $2 \times 3 = 6$ groups of division (b).

Division (c). $[(G^4{}_4 : G^5{}_{10})_2 : G^9{}_9]_1$. Each of the two groups $(G^4{}_4 : G^5{}_{10})_2$ can be multiplied directly by each of the two groups $G^9{}_9$. Hence there are $2 \times 2 = 4$ groups of division (c).

Division (d'). $[(G^9{}_{18} : G^5{}_{10})_2 : G^4{}_4]_2$. There are three groups $(G^9{}_{18} : G^5{}_{10})_2$. Each of these can be dimidiated in one way with each of the two groups $G^4{}_4$. Hence there are $3 \times 2 = 6$ groups of division (d').

Division (d''). $[(G^9_{18}: G^5_{10})_1: G^4_4]_4$. There are three groups G^9_{18} and one group G^5_{10}. Hence there are three groups of division (d'').

Divisions (e), (f) *and* (g). $[(G^5_{20}: G^4_4)_4: G^9_9]_1$, $[(G^5_{20}: G^4_4)_4: G^9_{18}]_2$ and $[(G^5_{20}: G^4_4)_4: G^9_{36}]_4$. There is only one group $(G^5_{20}: G^4_4)_4$. Hence there are two groups of division (e). This group of order 20 can be dimidiated in only one way, and hence yields with the three groups G^9_{18} the three groups of division (f). The only quotient group of order four in this group of order 20 is cyclic. As there is only one case of such a quotient group among the two groups G^9_{36}, and as this quotient group gives rise to only a single isomorphism, there is one group of division (g).

Divisions (h) *and* (i). $[(G^9_{36}: G^4_4)_4: G^5_5]_1$ and $[(G^9_{36}: G^4_4)_4: G^5_{10}]_2$. There are two groups $(G^9_{36}: G^4_4)_4$ which permit in all three distinct dimidiations. Thus with the one group G^5_5 they yield the two groups of division (h), and with the one group G^5_{10}, they yield the three groups of division (i).

Division (j). $[(G^4_{12}: G^9_{9k_2})_{3k_2}: G^5_{5k_3}]_{k_3}$. There is only one group G^4_{12}. It contains no invariant subgroup of order two and hence $k_2 \neq 2$. Neither group G^9_{36} contains a quotient group simply isomorphic with G^4_{12} and so $k_2 \neq 4$. Thus $k_2 = 1$ and a group of division (j) is $[(G^4_{12}: G^9_9)_3: G^5_{5k_3}]_{k_3}$. As neither of the two groups $(G^4_{12}: G^9_9)_3$ contains an invariant subgroup of index 2 or 4, $k_3 \neq 2$ or 4. Hence $k_3 = 1$ and there are two groups of division (j).

The number of groups of order 180 is thus:

Insolvable		1
Solvable	Division (a)	4
"	" (b)	6
"	" (c)	4
"	" (d')	6
"	" (d'')	3
"	" (e)	2
"	" (f)	3
"	" (g)	1
"	" (h)	2
"	" (i)	3
"	" (j)	2
Total		37

THE GROUPS OF ORDER $200 = 2^3 \cdot 5^2$.

Every group of order 200 is solvable and thus determines a $(G^8{}_{8k_1} : G^{25}_{25k_2})_{k_1k_2}$. $k_1 = 1$, as 8 is the only order which divides 200 for which transitive groups of degree 8 exist. $k_2 = 1$, 2, 4 or 8. Every group of order 200 occurs therefore in one of the following divisions:

Division (a) $(G^8{}_8 : G^{25}_{25})_1$ Division (c) $(G^8{}_8 : G^{25}_{100})_4$
Division (b) $(G^8{}_8 : G^{25}_{50})_2$ Division (d) $(G^8{}_8 : G^{25}_{200})_8$.

Division (a). $(G^8{}_8 : G^{25}_{25})$. A group in this division is the direct product of its Sylow subgroups. As there are five groups of order 8, and two groups of order 25, there are $5 \times 2 = 10$ groups of division (a). Six of these are abelian.

Division (b). $(G^8{}_8 : G^{25}_{50})$. The five groups of order 8 permit seven distinct dimidiations; the three groups G^{25}_{50} permit one dimidiation each. Hence there are $7 \times 3 = 21$ groups of division (b).

Division (c). $(G^8{}_8 : G^{25}_{100})$. There are six groups G^{25}_{100}. Five of them involve one case of cyclic quotient group of order four each: the sixth involves one case of non-cyclic quotient group of this order. The groups of division (c) may therefore be divided into two subdivisions.

(1) Quotient group of order 4 cyclic. The five groups of order 8 involve in all two cases of cyclic quotient group of order 4, and each case gives rise to only one isomorphism. Combination with the five groups G^{25}_{100} which involve cyclic quotient groups of order 4 therefore yields $2 \times 5 = 10$ groups of subdivision (1).

(2) Quotient group of order 4 non-cyclic. The one group G^{25}_{100} which involves a non-cyclic quotient group of order 4 contains just one characteristic subgroup of index 2. Hence there arise the following groups of subdivision (2).

Group of order 8	No. of groups of order 200
Cyclic	0
Abelian, type 2, 1....................	2
Dihedral	2
Dicyclic	1
Abelian, type 1, 1, 1..................	1
Total	6

Thus there are $10 + 6 = 16$ groups of division (c).

Division (d). ($G^8{}_8 : G^{25}_{200}$). A group of this division corresponds to a set of conjugate subgroups of order 8 in the i-group of a group of order 25. The i-group of the cyclic group of order 25 contains Sylow subgroups of order 4 and hence gives rise to no groups of this division. The i-group of the non-cyclic group of order 25 contains Sylow subgroups of order 32. The subgroups of order 8 are permuted under the i-group in five sets of conjugates, and thus there arise the five groups of division (d).

The number of groups of order 200 is thus:

Division	(a)	(b)	(c)	(d)	
Number	10	21	16	5	Total 52

There follows a list of the number of groups of every order (except 192) between 161 and 216 where this number exceeds one.

Order	Factors	Number of groups
162	pq^4	55
164	p^2q	5
165	pqr	2
166	pq	2
168	p^3qr	57
169	p^2	2
170	pqr	4
171	p^2q	5
172	p^2q	4
174	pqr	4
175	p^2q	2
176	p^4q	42
178	pq	2
180	p^2q^2r	37
182	pqr	4
183	pq	2
184	p^3q	12
186	pqr	6
188	p^2q	4
189	p^3q	13
190	pqr	4
192	p^6q	not determined
194	pq	2

Order	Factors	Number of groups
195	pqr	2
196	p^2q^2	12
198	pq^2r	10
200	p^3q^2	52
201	pq	2
202	pq	2
203	pq	2
204	p^2qr	12
205	pq	2
206	pq	2
207	p^2q	2
208	p^4q	51
210	$pqrs$	12
212	p^2q	5
214	pq	2

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A DETERMINATION OF ALL POSSIBLE SYSTEMS OF STRICT IMPLICATION.

By Morgan Ward.

1°. It is known that the postulates chosen by C. I. Lewis for his "system of strict implication" † are not categorical, since three distinct types of such a system have been shown to exist.‡ I shall prove here that the three types already discovered are the only ones possible. The inclusion of an additional modal postulate ‡ will therefore make the system categorical, and allow it to be exhibited as a four-valued truth-value system. The corresponding entscheidung problem may then be solved by the matrix method.

2°. In what follows, the decimal numeration 11.01-20.01 refers to *Symbolic Logic*, Chapter VI. We shall modify Lewis' notation as follows. We use $+$ instead of v to denote logical addition, p' for $\sim p$ and p^* for $\sim \Diamond p$. We shall refer to the system of strict implication as (the system) Σ.

Table I.

The System Σ.

Primitive Ideas

$p, p', \Diamond p, pq, p = q.$

Definitions

11.01 $p + q \cdot = \cdot (p'q')'$.
11.02 $p \prec q \cdot = \cdot (pq')^*$
11.03 $p = q \cdot = : p \prec q \cdot q \prec p$

Postulates

11.1 $pq \cdot \prec \cdot qp$
11.2 $pq \cdot \prec \cdot p$
11.3 $p \cdot \prec \cdot pp$
11.4 $(pq)r \cdot \prec \cdot p(qr)$
11.5 $p \cdot \prec \cdot (p')'$.
11.6 $p \prec q \cdot q \prec r : \prec \cdot p \prec r$
11.7 $p \cdot p \prec q : \prec q$
19.01 $\Diamond pq \cdot \prec \cdot \Diamond p$
20.01 $(\exists p, q) : (p \prec q)' \cdot (p \prec q')'$.

It is also assumed that the system is closed with respect to the unary operations $p', \Diamond p$ and the binary operation pq. The equality relation $=$ of the primitive ideas has the usual properties.§ In the present abstract treatment, 11.03 may be looked upon as a condition upon the relation $\prec$.

† It is assumed that the reader is familiar with the contents of Chapters VI and VII of C. I. Lewis and C. H. Langford's book, *Symbolic Logic* (New York, 1932), where a detailed account is given both of the system of strict implication and the matrix method as applied to truth-value systems. We shall refer to this book as *Symbolic Logic*.

‡ *Symbolic Logic*, Appendix II.

§ As given, for example, in E. V. Huntington's paper, "Postulates for the algebra of logic," *Transactions of the American Mathematical Society*, vol. 35 (1933), pp. 279-280.

3°. THEOREM.† *The system Σ is a Boolean algebra in which $p + q$ and pq are the operations of addition and multiplication, and p' is the negation of p.*

The following set of postulates for a Boolean algebra is given by Huntington in his Transactions paper, page 280. We presuppose a class K of elements $p, q, r, \cdots$ a unary operation p', a binary operation $+$ and an equality relation $=$ which we identify with the corresponding entities of Σ

H_0[20. 1, 20. 11] *K contains at least two distinct elements.*
H_2[11. 01] *If p and q are in the class K, then $p + q$ is in the class K.*
H_2[13. 11] $p + q = q + p$.
H_3[13. 4] $(p + q) + r = p + (q + r)$.
H_4[13. 31] $p + p = p$.
H_5[18. 2] $(p' + q')' + (p' + q)' = p$.
Def. H_6[11. 01, 12. 3] $pq = (p'q')'$.

The numbers in square brackets refer to the corresponding theorems in *Symbolic Logic*.

4°. THEOREM. *If the system of strict implication is interpreted as a truth-value system with a finite number of truth-values $n_1, n_2, \cdots, n_k$, then $n_1, n_2, \cdots, n_k$ must form a Boolean algebra $\mathfrak{B}$ with respect to the operations of addition, multiplication and negation derived from the matrices for $p + q$, pq and p'.*

For suppose that the matrices for p' and $p + q$ are

p	p'
n_1	β_1
n_2	β_2
.	.
.	.
.	.
n_k	β_k

$p \diagdown q$	n_1	n_2	$\cdots$	n_k
n_1	α_{11}	α_{12}	$\cdots$	α_{1k}
n_2	α_{21}	α_{22}	$\cdots$	α_{2k}
.	.	.	. . . .	.
.	.	.	. . . .	.
.	.	.	. . . .	.
n_k	α_{k1}	α_{k2}	$\cdots$	α_{kk}

where each α and β stands for a definite truth-value n. We then define the operations of negation and addition over $n_1, n_2, \cdots, n_k$ by

$$n'_i = \beta_i, \qquad n_i + n_j = \alpha_{ij} \qquad\qquad (i, j = 1, \cdots, k)$$

and it is immediately obvious that the conditions $H_0 - H_6$ of section 3° are all satisfied.

† For a detailed analysis of the correspondence between Σ and a Boolean algebra, see E. V. Huntington, *Bulletin of the American Mathematical Society*, vol. 40 (October, 1934), pp. 729-735.

COROLLARY. *The number of truth-values in any representation of Σ as a truth-value system is either infinite or a power of 2.*

Let us use the letters ∂ and ϵ to stand for designated values † and undesignated values in $\mathfrak{B}$ respectively. Then ∂ and ϵ combine in $\mathfrak{B}$ as follows:

TABLE II.

Combination of Truth-Values.

$+$	∂	ϵ
ϵ	∂	∂
∂	∂	ϵ

$\times$	∂	ϵ
∂	∂	ϵ
ϵ	ϵ	ϵ

$'$	
∂	ϵ
ϵ	∂

For example, the second table tells us that the product of two designated values is a designated value, the product of a designated value and an undesignated value is an undesignated value, and so on.

These facts result from the obvious propositions of Σ

$$p \cdot q : \prec : p + q \cdot pq; \quad pq' : \prec : p + q' \cdot (p'q)'; \quad p \cdot \prec \cdot (p')'.$$

5°. We consider now the possible representations of Σ as a four-valued truth-value system. In accordance with the results of section 4°, we may take for the set of truth-values $\mathfrak{B}$ the four numbers 1, 2, 3 and 6, which form a Boolean algebra if addition and multiplication are taken as the operations of finding the greatest common divisor and least common multiple, while negation is defined by $1' = 6$, $2' = 3$.

TABLE III.

Truth-Values of p', p^* and so on.

p	p'	p^*	$p + p'$	pp'	pp'^*	$\Diamond p$
1	6	a	1	6	d	a'
2	3	b	1	6	d	b'
3	2	c	1	6	d	c'
6	1	d	1	6	d	d'

There are in all $4^4 = 256$ such interpretations of Σ conceivable obtained by giving each of a, b, c, d, its four possible values 1, 2, 3, or 6. We shall use the definitions and postulates of Σ in Table I to reduce this number to eight.

From Table III, we see that ‡

(i) $d = \partial$, (ii) $6 \neq \partial$, (iii) $1 = \partial$.

† *Symbolic Logic*, pp. 231-233.

‡ We use the letter "∂" to stand for some designated value. Thus $6 \neq \partial$ means that 6 is not a designated value, and $ab, ac, ad = \partial$ would mean that ab, ac, and ad are all designated values.

From the last theorem of 4° and (ii) we see that

$$\text{(iv)}\quad \text{if } 2 = \partial,\ 3 \neq \partial;\ \text{if } 3 = \partial,\ 2 \neq \partial.$$

TABLE IV.

Matrices for pq, pq' and so on.

pq	pq'	$p \prec q$	$q \prec p$			$p = q$	
1 2 3 6	6 3 2 1	$d\ c\ b\ a$	$d\ d\ d\ d$	d	$d\ c$	$a\ b$	$d\ a$
2 2 6 6	6 6 2 2	$d\ d\ b\ b$	$c\ d\ c\ d$	$d\ c$	d	$b\ c$	$b\ d$
3 6 3 6	6 3 6 3	$d\ c\ d\ c$	$b\ b\ d\ d$	$a\ b$	$b\ c$	d	$d\ c$
6 6 6 6	6 6 6 6	$d\ d\ d\ d$	$a\ b\ c\ d$	$d\ a$	$b\ d$	$d\ c$	d

Now since equality over Σ is defined as logical equivalence,† $p = q$ when and only when p and q have the same truth-values. Therefore, we infer from the matrix for $p = q$ that $ad, bc, bd, cd \neq \partial$. Hence by (i) and Table II,

$$\text{(v)}\quad a, b, c \neq \partial.$$

From (v), (i) and (iii), we see that

$$\text{(vi)}\quad a, b, c \neq d \qquad\qquad \text{(vii)}\quad a, b, c \neq 1.$$

TABLE V.

The Principle of the syllogism.

p	q	q'	$p \prec q$	$p \cdot p \prec q$	$p \cdot p \prec q : q'$	$11.7 \cdot = \cdot (p \cdot p \prec q : q')^*$
1	1	6	d	d	6	d
1	2	3	c	c	$3c$	$(3c)^*$
1	3	2	b	b	$2b$	$(2b)^*$
1	6	1	a	a	a	a^*
2	1	6	d	$2d$	6	d
2	2	3	d	$2d$	6	d
2	3	2	b	$2b$	$2b$	$(2b)^*$
2	6	1	b	$2b$	$2b$	$(2b)^*$
3	1	6	d	$3d$	6	d
3	2	3	c	$3c$	$3c$	$(3c)^*$
3	3	2	d	$3d$	6	d
3	6	1	c	$3c$	$3c$	$(3c)^*$
6	1	6	d	6	6	d
6	2	3	d	6	6	d
6	3	2	d	6	6	d
6	6	1	d	6	6	d

† Lewis and Langford, pp. 123-124.

From the last column of Table V, we see that

$$\text{(viii)} \quad a^*, (2b)^*, (3c)^* = \partial.$$

I say that $a = 6$. For by (vii), $a \neq 1$. And if $a = 2$ or 3, by (viii), $a^* = 2^*$ or $a^* = 3^*$. Hence $a^* = b$ or $c, = \partial$ contradicting (v).

I say that $b = 3$ or $b = 6$. For by (vii), $b \neq 1$. And if $b = 2$, then by (viii), $(2b)^* = 2^* = b = 2 = \partial$ contradicting (v).

Finally, $c = 2$ or $c = 6$. For by (vii), $c \neq 1$. And if $c = 3$, then by (viii) $(3c)^* = 3^* = c = 3 = \partial$ contradicting (v).

We cannot have $b = 3$ and $c = 2$. For then $d = 1$ by (ii) and (v). Hence $\Diamond p \cdot = \cdot p'$ and Σ will degenerate into a system of material implication, contradicting 20. 01.

We summarize our results in the following

THEOREM. *There are at most eight possible four-valued systems of strict implication, distinguished by the truth-values of* $\Diamond p$; *namely*

TABLE VI.
Possible Systems Σ.

p	$\Diamond p$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1		1	1	1	1	1	1	1	1
2		1	1	2	2	1	1	1	1
3		3	3	1	1	1	1	1	1
6		6	2	6	3	6	6	3	2
Designated Values †		1, 3	1, 3	1, 2	1, 2	1, 2	1, 2	1, 2	1, 3

These systems may be grouped into four pairs, (7) and (8); (1) and (3); (5) and (6); (2) and (4); which are permuted into one another by the interchange of the truth-values 2 and 3, and are hence not essentially distinct. Finally, the four pairs are immediately seen to agree with the systems called Group I, Group II, Group III and Group V, in Appendix II of *Symbolic Logic.*

I have verified that the first three pairs satisfy all the postulates of Σ, while the last pair satisfy all the postulates save 19. 01, as was first proved by W. T. Parry, M. Wajsberg and P. Henle.‡ I shall denote these three systems of strict implication by Σ_1, Σ_2, Σ_3.

† Obtained by (i), (ii) and (iv).

‡ *Symbolic Logic,* footnote, page 492.

6°. It remains to show that there is no representation of Σ as a truth-value system of finite order † essentially distinct from Σ_1, Σ_2 and Σ_3.

Suppose that a representation of Σ as a truth-value system maps Σ upon a Boolean algebra $\mathfrak{B}_N$ of order 2^N, $N \geqq 3$ such that all the postulates of Σ are satisfied in accordance with the matrix method.

Let N generating elements of the algebra $\mathfrak{B}_N$ be $\alpha_1, \alpha_2, \cdots, \alpha_N$. Since $N \geqq 3$, we see from Table II that there are at least two generators which are both designated values, or at least two generators which are undesignated values. With a proper choice of notation, we may assume that α_1, α_2 are such a pair.

Now every element ν of the algebra $\mathfrak{B}_N$ may be uniquely represented in the form

(1) $$\nu = \alpha_1^{e_1}\alpha_2^{e_2} \cdots \alpha_N^{e_N}$$

where the exponents e are either zero or one, and by convention, the universal element of $\mathfrak{B}_N$ is denoted by 1, $\alpha^0 = 1$.

Consider now the effect of equating α_1 and α_2. An inspection of Table II and (1) shows us that this operation does not convert any designated value into an undesignated value, or vice versa. Hence the truth-value table establishing the validity of any one of our postulates for Σ in $\mathfrak{B}_N$, is unaffected by the operation.‡

This operation, however, throws $\mathfrak{B}_N$ into a Boolean algebra $\mathfrak{B}_{N-1}$ of order 2^{N-1} on which Σ is, therefore, mapped. On repeating this process $N - 2$ times, we obtain a mapping upon the Boolean algebra $\mathfrak{B}_2$. On retracing our steps from $\mathfrak{B}_2$ to $\mathfrak{B}_3$ to $\mathfrak{B}_4$ and so on to $\mathfrak{B}_N$, we see that we have a multiple isomorphism between $\mathfrak{B}_N$ and $\mathfrak{B}_2$ which preserves the assertion values of all the postulates for Σ. Hence, the mapping on $\mathfrak{B}_N$ is not essentially distinct from one of the three possible mappings on $\mathfrak{B}_2$.

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† The question of whether representations of Σ as a truth-value system of infinite order exist is left open.

‡ The reader may find it helpful to glance back at Table V. In the mapping over $\mathfrak{B}_N$, 1, 2, 3, 6, will be replaced by the 2^N elements of $\mathfrak{B}_N$. However, the elements on the extreme right of Table V which are all designated values of $\mathfrak{B}_N$, will remain designated values after equating α_1 and α_2.

ON THE PROGRESSIONS ASSOCIATED WITH A TERNARY QUADRATIC FORM.

By E. H. Hadlock.

1. *Introduction.* Denote the primitive ternary quadratic form $ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy$ by f, its reciprocal by F, its Hessian or determinant by H, $(H \neq 0)$, and the greatest common divisor of the cofactors of a, b, c etc., in H by Ω. Then Δ is defined by $H = \Omega^2\Delta$.

B. W. Jones * has shown that with every ternary quadratic form f of Hessian H there is associated a set of arithmetic progressions:

$$(1) \qquad 2^r(8n + a'_j), \qquad p_i^{r_i}(p_i n + a_{ij}) \qquad (n = 0, \pm 1, \pm 2, \cdots)$$

such that no integer falling in any one of them is represented by f, and for every integer a not falling in any of them it is true that $f \equiv a \pmod N$, for N arbitrary,† is solvable, where p_i are odd prime factors of H, a_{ij} are some or all the members of a complete residue system mod p_i, r and r_i range over some or all of the positive integers and zero, and a'_j are some, none or all of $1, 3, 5, 7$.

But in this paper we will speak of $2^r(8n + a'_j)$ as *a set of progressions* associated with f where a'_j is one of $1, 3, 5$, or 7. Similarly, $p_i^{r_i}(p_i n + a_{ij})$ will be a set for each p_i of H.

In Art. I it is shown that Ω, Δ together with the order and the generic characters as defined by H. J. S. Smith ‡ determine the progressions associated with a given form; and conversely, that Ω, Δ and the progressions associated with a given form determine the generic characters. In fact, it is important to notice that Ω and Δ restrict the choice of the progressions (1) as is seen on pp. 103-109. We shall speak of the progressions (1) associated with a given form as progressions corresponding to the generic characters and the invariants Ω and Δ of the form, or simply *corresponding progressions.* The corresponding progressions are given on pp. 103-109.

Smith § has shown that there exists a properly primitive form f having

* B. W. Jones, "A new definition of genus for ternary quadratic forms," *Transactions of the American Mathematical Society*, vol. (1931), No. 1, pp. 92-110. This article will be referred to as Art. I.

† This condition implies that a is represented by some form of the same genus as f. (See B. W. Jones, "Regularity of a genus of positive ternary quadratic forms," *Transactions of the American Mathematical Society*, vol. 33 (1931), No. 1, pp. 111-124.)

‡ H. J. S. Smith, *Collected Mathematical Papers*, vol. 1, pp. 457-459; L. E. Dickson, "Studies in the Theory of Numbers," pp. 51, 52.

§ H. J. S. Smith, *loc. cit.*, p. 470.

a given Ω and Δ, a given set of values for the generic characters and whose reciprocal F is also properly primitive if and only if

$$\psi g^{e_f} G^{e_F} (f \mid \Omega_1)(F \mid \Delta_1) = (-1)^{e_2}, \tag{2}$$

where

$$\begin{array}{llll} g=1 & \text{if} \quad \Omega=\Omega_1\Omega_2^2, & g=-1 & \text{if} \quad \Omega=2\Omega_1\Omega_2^2, \\ G=1 & \text{if} \quad \Delta=\Delta_1\Delta_2^2, & G=-1 & \text{if} \quad \Delta=2\Delta_1\Delta_2^2, \\ e_2=(\Omega_1+1)(\Delta_1+1)/4, & & e_P=(P^2-1)/8, & \end{array} \tag{2a}$$

and $\Omega_2{}^2$ and $\Delta_2{}^2$ are the largest squares dividing Ω and Δ respectively. $\Omega=\Omega_1\Omega_2{}^2$ or $2\Omega_1\Omega_2{}^2$ according as $\Omega/\Omega_2{}^2$ is odd or even; similarly for Δ. Hence Ω_1 and Δ_1 are always odd and not divisible by any square. If f is improperly primitive then instead of (2) the condition is

$$(-G)^{e_F}(2f \mid \Omega_1)(F \mid \Delta_1) = (-1)^{e_2}. \tag{3}$$

If F is improperly primitive, the condition is

$$(-g)^{e_f}\ (f \mid \Omega_1)(2F \mid \Delta_1) = (-1)^{e_2}. \tag{4}$$

The purpose of this paper is to find conditions on the progressions associated with a form which are equivalent to Smith's character conditions (2)-(4). This leads to the fact that the number of sets of corresponding progressions of a certain kind is odd or even according as f is positive or indefinite. (See Theorem II of this paper). It is also found that with every positive form there are associated infinitely many progressions of numbers not represented by f.

2. From (2a) we notice that the odd primes which occur to even powers in both Ω and Δ do not affect the value of $(f \mid \Omega_1)(F \mid \Delta_1)$ in (2)-(4). Then, suppose we have given Ω, Δ and only the following sets of corresponding progressions I-X involving the distinct odd prime factors $p_1, p_2, \cdots, p_n$ which occur to odd powers in at least one of Ω and Δ. From Art. I, p. 103, it is seen that if we omit the progressions $p_a{}^{2s+1}$ then I-X include all combinations of sets of corresponding progressions in $p_1, p_2, \cdots, p_n$.

I. $p_{1j}{}^{2k}(p_{1j}n+\alpha_{1j})$,

II. $p_{2j}{}^{2k}(p_{2j}n+\alpha_{2j})$, $p_{2j}{}^{2s_{2j}+1}(p_{2j}n+\beta_{2j})$,

III. $p_{3j}{}^{2r_{3j}}(p_{3j}n+\alpha_{3j})$,

IV. $p_{4j}{}^{2r_{4j}}(p_{4j}n+\alpha_{4j})$, $p_{4j}{}^{2s_{4j}+1}(p_{4j}n+\beta_{4j})$,

V. $p_{5j}{}^{2k+1}(p_{5j}n+\beta_{5j})$, $p_{5j}{}^{2r_{5j}}(p_{5j}n+\alpha_{5j})$,

VI. $p_{6j}{}^{2r_{6j}}(p_{6j}n+\alpha_{6j})$, $p_{6j}{}^{2s_{6j}+1}(p_{6j}n+\beta_{6j})$,

VII. $p_{7j}{}^{2k+1}(p_{7j}n+\beta_{7j})$,

VIII. $p_{8j}{}^{2k+1}(p_{8j}n+\beta_{8j})$, $p_{8j}{}^{2r_{8j}}(p_{8j}n+\alpha_{8j})$,

IX. $p_{9j}{}^{2r_{9j}}(p_{9j}n+\alpha_{9j})$,

X. none for $p_{10,j}$,

where $(j=1,2,\cdots,N_i)$, $(i=1,2,\cdots,10)$ and the p_{ij}'s are $p_1,p_2,\cdots,p_n$ renamed. Each α_{ij} and β_{ij} represents all the quadratic residues or all the quadratic non-residues of p_{ij}; the ranges r_{ij} and s_{ij} are finite and that of k is infinite. In Art. I, p. 103, α_{-1}, α_1, s, s_1, r, k, Ω', Δ' correspond to

(5) $$\alpha_{ij},\ \beta_{ij},\ s'_{ij},\ s_{ij},\ r_{ij},\ k,\ \Omega'_{ij},\ \Delta'_{ij},$$

respectively. Ω'_{ij} and Δ'_{ij} are defined by

(6) $$(\Omega/p_{ij}^{t_{ij}})=\Omega'_{ij}\not\equiv 0 \pmod{p_{ij}},\qquad (\Delta/p_{ij}^{t'_{ij}})=\Delta'_{ij}\not\equiv 0 \pmod{p_{ij}}.$$

In I-IV, p_{ij} occurs to odd and even powers in Ω and Δ respectively; in V and VI, p_{ij} occurs to odd powers in both Ω and Δ; and in VII-X, p_{ij} occurs to an odd power in Δ and to an even power in Ω. p_{ij} of I and III is not a factor of Δ and p_{ij} of VII and X is not a factor of Ω.

Define

(7) $$\Omega_{ii}=\prod_{j=1}^{N_i} p_{ij}\ (i=1,2,\cdots,6),\qquad \Delta_{ii}=\prod_{j=1}^{N_i} p_{ij}\ (i=5,\cdots,10),$$

(8) $$A=\prod_{i=1}^{4}\Omega_{ii},\qquad B=\prod_{i=7}^{10}\Delta_{ii},$$

(9) $$C=\Omega_{55}\Omega_{66}=\Delta_{55}\Delta_{66},$$

(10) $$J(u,v,w)=(-1\mid uvw)(u\mid v)(v\mid u)(uv\mid w)(w\mid uv),$$

(11) $$e_3=N_1+N_2+N_5+N_7+N_8.$$

If $N_i=0$, define $\Omega_{ii}=1$, $(i=1,2,\cdots,6)$ and $\Delta_{ii}=1$, $(i=5,\cdots,10)$.

From (2a), (7), (8), and (9), we have

(12) $$|\,\Omega_1\,|=\prod_{i=1}^{6}\Omega_{ii}=AC,\qquad |\,\Delta_1\,|=\prod_{i=5}^{10}\Delta_{ii}=BC.$$

Case 1. Ω and Δ are each positive.

Hence $\Omega_1=AC$ and $\Delta_1=BC$. From Lemma 12 of Art. I we notice that

(13) $$(f\mid p)=(\alpha\mid p)$$

for each p of Ω and from the corollary of Lemma 4 we notice also that

(14) $$(\alpha\mid p)=-(\alpha_{-1}\mid p).$$

With the aid of (13), (14), (5), and (7) we obtain

(15) $$(f\mid\Omega_{ii})=(-1)^{N_i}\prod_{j=1}^{N_i}(\alpha_{ij}\mid p_{ij}),\qquad (i=1,2,\cdots,6).$$

From Art. I, p. 103 we have for each p in I-II, III-IV, VII-VIII, IX-X, V and VI respectively, the conditions

$$(16) \qquad (-\alpha\Delta' \mid p) = -1, \qquad (-\alpha\Delta' \mid p) = 1,$$

$$(17) \qquad (F \mid p) = -(-\Omega' \mid p), \qquad (F \mid p) = (-\Omega' \mid p),$$

$$(18) \qquad (F \mid p) = -(-\alpha\Delta'\Omega' \mid p), \qquad (F \mid p) = (-\alpha\Delta'\Omega' \mid p).$$

From (15) we have with the aid of (16), (6), and (2a)

$$(19) \qquad (f \mid \Omega_{ii}) = \gamma^{N_i} G^{e\Omega_{ii}} (-\Delta_1 \mid \Omega_{ii}), \qquad (i = 1, 2, 3, 4),$$

where $\gamma = -1$ if $(i = 1, 2)$, $\gamma = 1$ if $(i = 3, 4)$. From (17) we have

$$(20) \qquad (F \mid \Delta_{ii}) = \gamma^{N_i} g^{e\Delta_{ii}} (-\Omega_1 \mid \Delta_{ii}), \qquad (i = 7, \cdots, 10),$$

where $\gamma = -1$ if $(i = 7, 8)$, $\gamma = 1$ if $(i = 9, 10)$. From (8), (9), and (12) we have

$$(21) \qquad \begin{aligned} (\Delta'_{5j} \mid p_{5j}) &= G^{e_{p_{5j}}} (B\Delta_{66} \mid p_{5j}) (p_{51}, p_{52}, \cdots, p_{5,j-1}, p_{5,j+1}, \cdots, p_{5N_5} \mid p_{5j}), \\ (-\Omega'_{5j} \mid p_{5j}) &= g^{e_{p_{5j}}} (-A\Delta_{66} \mid p_{5j}) (p_{51}, p_{52}, \cdots, p_{5,j-1}, p_{5,j+1}, \cdots, p_{5N_5} \mid p_{5j}). \end{aligned}$$

Then from (18) and (21) we obtain

$$(22) \qquad (F \mid \Delta_{55}) = (gG)^{e\Delta_{55}} (-AB \mid \Delta_{55}) \prod_{j=1}^{N_5} (\alpha_{5j} \mid p_{5j}).$$

Similarly we obtain

$$(23) \qquad (F \mid \Delta_{66}) = (-1)^{N_6} (gG)^{e\Delta_{66}} (-AB \mid \Delta_{66}) \prod_{j=1}^{N_6} (\alpha_{6j} \mid p_{6j}).$$

From (15) and (19) we have

$$(24) \quad (f \mid \Omega_1) = (-1)^{e_4} G^{eA} (-1 \mid A) (\Delta_1 \mid A) \prod_{j=1}^{N_5} (\alpha_{5j} \mid p_{5j}) \prod_{j=1}^{N_6} (\alpha_{6j} \mid p_{6j}),$$

where $e_4 = N_1 + N_2 + N_5 + N_6$. From (20), (22) and (23) we obtain

$$(25) \quad (F \mid \Delta_1) \\ = (-1)^{e_5} g^{e\Delta_1} G^{eC} (-1 \mid BC) (\Omega_1 \mid B) (AB \mid C) \prod_{j=1}^{N_5} (\alpha_{5j} \mid p_{5j}) \prod_{j=1}^{N_6} (\alpha_{6j} \mid p_{6j})$$

where $e_5 = N_6 + N_7 + N_8$. Then from (24), (25), (10) and (11) we have

$$(26) \qquad (f \mid \Omega_1)(F \mid \Delta_1) = J(A, B, C)(-1)^{e_3} g^{e\Delta_1} G^{e\Omega_1}.$$

For the two cases, $A \equiv B \pmod 4$, and $A \equiv 3B \pmod 4$

$$(27) \qquad J(A, B, C)(-1)^{e_2} = -1.$$

Case 2. Ω and Δ have opposite signs.

If $\Omega < 0$, then from (2a) we notice that $\Omega_1 < 0$. Then in (12) we take $\Omega_1 = (-A)C$. Instead of $J(A, B, C)$ in (26) we have $J(-A, B, C)$. But

$$(28) \qquad J(-A, B, C)(-1)^{e_2} = 1.$$

If $\Omega > 0$ we also have $J(A, -B, C)(-1)^{e_2} = 1$.

Replace $(f \mid \Omega_1)(F \mid \Delta_1)$ in (2) by its value in (26) and use (27) and (28). We obtain

$$(29) \qquad \psi g^{e_{\Delta_1 f}} G^{e_{\Omega_1 F}} = R(-1)^{e_3}$$

where $R = -1$ if Ω and Δ are each positive, and $R = 1$ if Ω and Δ have opposite signs. From the progressions I-X we notice that e_3 is the number of sets of corresponding progressions of the type $p_{ij}^{k_1}(p_{ij}n + C_{ij})$ where $C_{ij} = \alpha_{ij}$ or β_{ij} and $k_1 = 1, 3, 5 \cdots$ or $k_1 = 0, 2, 4, \cdots$; that is, the range of the exponent k_1 of p_{ij} is infinite. The condition (29) is equivalent to (2).

When we apply to (29) each of the cases A-F * in Art. I, pp. 104-108, we find that there exists a positive form or an indefinite form if and only if the number of sets of corresponding progressions

$$(30) \qquad p_{ij}^{k_1}(p_{ij}n + C_{ij}), \qquad 2^{k_2}(8n + a'_j)$$

is respectively odd or even, where k_i $(i = 1, 2)$ ranges over all the evens or over all the odds. In particular we find from A that an indefinite form may have no progressions at all associated with it.

As an illustration of the application of A-E to (29) we take in E $(\Omega'' = 8, \Delta'' = 2)$ the progressions $4n + \Delta', 4n + 2, 8n + 5\alpha\Delta', 4^k(16n + 14\Delta')$ which are found in the last row, with $\alpha \equiv 3 \pmod 4$, and $\Omega_1 F \equiv \Omega' F \equiv 3\alpha, 5\alpha \pmod 8$. From (29) we have $(2 \mid \Omega_1 F)(2 \mid \Delta_1 f)\psi = R(-1)^{e_3}$. If $\alpha \equiv 3 \pmod 8$, then $f \equiv 3\Delta' \pmod 8$; hence, $(2 \mid \Delta_1 f) = -1$, and $\psi = 1$. Also $(2 \mid \Omega_1 F) = 1$ since $\Omega_1 F \equiv 1, 7 \pmod 8$. Then $R(-1)^{e_3} = -1$, and e_3 is even or odd according as $R = -1$ or 1. Similarly, if $\alpha \equiv 7 \pmod 8$.

For f in (3) $\Omega \equiv 1 \not\equiv \Delta \pmod 2$. Now if we replace $(f \mid \Omega_1)(F \mid \Delta_1)$ in (3) by its value given in (26) we find that

$$(31) \qquad \begin{aligned} R(-1)^{e_3} &= 1 \quad \text{if} \quad G = -1, \\ R(-1)^{e_3} &= (2 \mid \Omega_1 F) \quad \text{if} \quad G = 1. \end{aligned}$$

Then, from Art. I, p. 109, G, we find again that there exists a positive form or an indefinite form if and only if the number of sets of corresponding progressions (30) is respectively odd or even.

For f in (4) $\Delta \equiv 1 \not\equiv \Omega \pmod 2$. Now if we replace $(f \mid \Omega_1)(F \mid \Delta_1)$ in (4) by its value given in (26) we find that

* $2^{k_2}(8n + a'_j)$ may be obtained directly from the congruences in III of Art. I. p. 104.

$$(32)\qquad \begin{aligned} R(-1)^{e_3} &= 1 \quad \text{if} \quad g = -1, \\ R(-1)^{e_3} &= (2 \mid \Delta_1 f) \quad \text{if} \quad g = 1. \end{aligned}$$

If we apply the principle of reciprocity to (32) we again obtain (31). If now we display the progressions $4^k(8n + a'_j)$ of F in Art. I, p. 108, G, and apply (31) to them we find that we can have a form F and hence f having the corresponding progressions (30) if and only if the number of their sets is odd or even according as F is positive or indefinite.

Since (29), (31), and (32) are equivalent to (2), (3), and (4) respectively, we have

THEOREM I. *Each of Smith's character conditions* (2)-(4) *is equivalent to the fact that the number of sets of corresponding progressions * of the type* $p_{ij}^{k_1}(p_{ij}n + C_{ij})$, $2^{k_2}(8n + a'_j)$ *in the progressions* (1) *is odd or even according as* Ω *and* Δ *have the same or opposite signs, and where* $C_{ij} = \alpha_{ij}$ *or* β_{ij} *and* k_i $(i = 1, 2)$ *ranges over all the evens or over all the odds.*

THEOREM II. *There is an odd (even) number of sets of progressions of the type* $p_{ij}^{k_1}(p_{ij}n + C_{ij})$, $2^{k_2}(8n + a'_j)$ *in the progressions* (1) *associated with a positive (indefinite) form where* $C_{ij} = \alpha_{ij}$ *or* β_{ij} and k_i $(i = 1, 2)$ *ranges over all the evens or over all the odds. Conversely; if* Ω *and* Δ *are given and if there is given an odd (even) number of sets of corresponding progressions of the type* $p_{ij}^{k_1}(p_{ij}n + C_{ij})$, $2^{k_2}(8n + a'_j)$ *in the progressions* (1), *then there exists a positive (indefinite) form associated with the given progressions.*

COROLLARY. *With every positive ternary quadratic form there are associated infinitely many progressions* (1).

By inspection of p. 103 of Art. I we have the following properties: $P_1 - P_5$ where $L_{ij} = [(t_{ij} - 2)/2]$, $M_{ij} = [L_{ij} + t'_{ij}/2]$ and t_{ij}, t'_{ij} are defined in (6). $[-1/2] = -1$.

P_1. If $t_{ij} \geqq 2$ is even then $r_{ij} = 0, 1, 2, \cdots, L_{ij}$, if t_{ij} is odd then $r_{ij} = 0, 1, 2, \cdots, M_{ij} + 1$.

P_2. If $t_{ij} \geqq 2$ then $s'_{ij} = 0, 1, 2, \cdots, L_{ij}$ unless $p_{ij}^{2k+1}(p_{ij}n + \beta_{ij})$ occurs when t_{ij} is even. If t_{ij} is even and $t_{ij} + t'_{ij} \geqq 2$ then $s'_{ij} = 0, 1, 2, \cdots, M_{ij}$.

P_3. If $t_{ij} \geqq 1$ is odd then $s_{ij} = 0, 1, 2, \cdots, M_{ij}$ or $M_{ij} + 1$ according as t'_{ij} is even or odd.

P_4. If t'_{ij} is odd then $(\beta_{ij} \mid p_{ij}) = (-\Delta'_{ij} \mid p_{ij})$ or $-(-\Delta'_{ij} \mid p_{ij})$ according as $p_{ij}^{2k+1}(p_{ij}n + \beta_{ij})$ or $p_{ij}^{2s_{ij}+1}(p_{ij}n + \beta_{ij})$ occurs; otherwise the value of $(\beta_{ij} \mid p_{ij})$ can be chosen arbitrarily.

* When F is improperly primitive the progressions (30) of f can readily be found.

P_5. If t_{ij} is odd and t'_{ij} is even, then $(\alpha_{ij} \mid p_{ij}) = (-\Delta_{ij} \mid p_{ij})$ or $-(-\Delta_{ij} \mid p_{ij})$ according as $p_{ij}^{2k}(p_{ij}n + \alpha_{ij})$ on $p_{ij}^{2r_{ij}}(p_{ij}n + \alpha_{ij})$ occurs; otherwise the value of $(\alpha_{ij} \mid p_{ij})$ can be chosen arbitrarily.

3. *Example.* We now give an example of the converse part of Theorem II. Suppose we have given $\Omega = 6$, $\Delta = 105$. Since 3 occurs to an odd power in both Ω and Δ, we may choose either V or VI. Choose $3^{2k+1}(3n + \beta_{51})$, $3^{r_{51}}(3n + \alpha_{51})$. From P_4, $(\beta_{51} \mid 3) = (-35 \mid 3) = 1$. From P_1 we notice that $r_{51} = 0$ and from P_5 we may choose $\alpha_{51} \equiv 2 \pmod 3$. Since 5 is not a factor of Ω and occurs to an odd power in Δ we may choose either VII or X. Choose 5^{2k+1} $(5n + \beta_{71})$. $(\beta_{71} \mid 5) = (-21 \mid 5) = 1$. Let there be no progressions involving 7. There are no progressions $p^{2s+1}a$ for $p = 3$, 5, and 7. The number of sets of progressions $3^{2k+1}(3n + 1)$ and $5^{2k+1}(5n \pm 1)$ is even. Hence in Art. I, p. 104 ($\Omega'' = 2$, $\Delta'' = 1$) we must take $4^k(8n + 7\Delta')$. $\Delta' = \Delta \equiv 1 \pmod 8$.

It remains to find a form f with $\Omega = 6$, $\Delta = 105$, and having the progressions $3^{2k+1}(3n + 1)$, $3n + 2$, $5^{2k+1}(5n \pm 1)$ and $4^k(8n + 7)$ associated with it.

From $f = ax^2 + by^2 + cz^2 + 2ryz + 2xz$ we obtain $abf = bx_1^2 + ay_1^2 + Hz^2$ where $x_1 = ax + z$, $y_1 = by + rz$ and $H = a\Omega A - b$. Hence $b \equiv 0 \pmod \Omega$. Take $b = 6b'$ with b' prime to 6. $g = 2ab'f = 2b'x_1^2 + 3ay_2^2 + 1260z^2$ where $y_1 = 3y_2$. In order for f to have the progression $3n + 2$ associated with it, we take $a \equiv 1 \pmod 3$. $g \equiv 0 \pmod 3$ implies that $x_1 = 3x_2$. Then $g/3 = 6b'x_2^2 + ay_2^2 + 420z^2$. We take $b' \equiv 1 \pmod 3$. Then f will have the progressions $3^{2k+1}(3n + 1)$ associated with it as is seen from the corollary of Lemma 5 of Art. I. Similarly, we take $(ab' \mid 5) = -1$ and $(ab' \mid 7) = 1$. Let $a = 1$. $b'f = b'x_1^2 + 6y_3^2 + 630z^2$ where $y_2 = 2y_3$. Let $b' \equiv 3 \pmod 8$. Then from the corollary of Lemma 11 of Art. I, f will have the progressions $4^k(8n + 7)$ associated with it. Take $b' = 67$. Then $b = 402$. From $H = a(bc - r^2) - b$ we have $c = 133$, and $r = 222$. Hence $f = x^2 + 402y^2 + 133z^2 + 444yz + 2xz$. From $x_1 = x + z$, x will be an integer for any integers assigned to x_1 and z. $y_1 = by + rz = 6y_3$ where $y_3 = 67y + 37z$. $b'f \equiv 0 \pmod{67}$ implies that $y_3 \equiv \pm 37z \pmod{67}$. Hence the sign of z can be so chosen that y is an integer. Apply to f the transformation $x = x' + 2y' + z'$, $y = y' + z'$ and $z = -2y' - z'$ of determinant 1. We find that f is equivalent to the reduced form $x'^2 + 42y'^2 + 90z'^2$.

4. *Table.* In the following table * abbreviate the form f by enclosing the

* Eisenstein has given a table of genera for forms with odd Hessian from 1 to 25, "neue Theoreme der höheren Arithmetik," *Journal für Mathematik*, vol. 35 (1847), p. 136.

coefficients of the square terms and half the coefficients of the product terms in parentheses: (a, b, c, r, s, t). Let P denote the progressions (1) associated with a form f. Let $s_1 = (2 \mid f)\psi$ and $s_2 = (2 \mid f)(2 \mid F)\psi$. An asterisk prefixed to a form indicates that f has an improperly primitive reciprocal F. If f is improperly primitive, then f always has the progression $2n + 1$ associated with it. The progression $2n + 1$ is not written in the table.

TABLE OF GENERIC CHARACTERS AND PROGRESSIONS OF REDUCED POSITIVE TERNARY QUADRATIC FORMS FOR TYPICAL VALUES OF H FROM 1 TO 25.

f Properly Primitive.

H odd.

H	$(F \mid p)$ or $(f \mid p)$	ψ	P	Forms
1		-1	$4^k(8n+7)$	$(1, 1, 1, 0, 0, 0)$
3	1	1	$3^{2k+1}(3n+2)$	$(1, 1, 3, 0, 0, 0)$
3	-1	-1	$4^k(8n+5)$	$(1, 2, 2, -1, 0, 0)$
5	1	-1	$4^k(8n+3)$	$(1, 1, 5, 0, 0, 0)$
5	-1	1	$5^{2k+1}(5n \pm 1)$	$(1, 2, 3, -1, 0, 0)$
9	1	-1	$4^k(8n+7)$	$(1, 1, 9, 0, 0, 0)$
			$3(3n \pm 1)$	
9	-1	-1	$4^k(8n+7)$	$(1, 2, 5, -1, 0, 0)$
9	$(f \mid 3) = 1$	1	$3^{2k}(3n+2)$	$(1, 3, 3, 0, 0, 0)$
9	$(f \mid 3) = -1$	-1	$4^k(8n+7)$	$(2, 2, 3, 0, 0, -1)$
			$3n+1$	
15	1	1	$3^{2k+1}(3n+1)$	$(1, 1, 15, 0, 0, 0)$
	1			$(1, 4, 4, -1, 0, 0)$
15	-1	1	$5^{2k+1}(5n \pm 2)$	$(1, 2, 8, -1, 0, 0)$
	-1			$(1, 3, 5, 0, 0, 0)$
15	$(F \mid 3) = 1$	-1	$4^k(8n+1)$	$(2, 2, 5, 0, 0, -1)$
	$(F \mid 5) = -1$		$5^{2k+1}(5n \pm 2)$	
			$3^{2k+1}(3n+1)$	
15	$(F \mid 3) = -1$	-1	$4^k(8n+1)$	$(2, 3, 3, 0, 0, -1)$
	$(F \mid 5) = 1$			

$H \equiv 2 \pmod 4$

H	$(F \mid p)$ or $(f \mid p)$	$(2 \mid F)\psi$	P	Forms
2		-1	$4^k(16n+14)$	$(1, 1, 2, 0, 0, 0)$
6	1	1	$3^{2k+1}(3n+1)$	$(1, 1, 6, 0, 0, 0)$
6	-1	-1	$4^k(16n+10)$	$(1, 2, 3, 0, 0, 0)$
18	1	-1	$4^k(16n+14)$	$(1, 1, 18, 0, 0, 0)$
			$3(3n \pm 1)$	$(2, 2, 5, 0, -1, 0)$
18	-1	-1	$4^k(16n+14)$	$(1, 2, 9, 0, 0, 0)$
				$(2, 3, 4, -1, 0, -1)$
18	$(f \mid 3) = 1$	1	$4^k(16n+14)$	$(1, 3, 6, 0, 0, 0)$
			$3n+2$	
18	$(f \mid 3) = -1$	-1	$3^{2k}(3n+1)$	$(2, 3, 3, 0, 0, 0)$

$H \equiv 0 \pmod 4$

H	$(F \mid p)$	$f \equiv$ (mod 8) or s_1	$F \equiv$ (mod 8) or s_2	P	Forms
4		1, 5	1, 5	$4^k(8n+7)$ $4n+3$	(1, 1, 4, 0, 0, 0)
4		$s_1 = -1$		$4^k(8n+7)$	(1, 2, 2, 0, 0, 0)
8		1, 5	1	$4^k(16n+14)$ $8n+6$ $4n+3$	(1, 1, 8, 0, 0, 0)
8			$s_2 = -1$	$4^k(16n+14)$	(1, 2, 4, 0, 0, 0)
8			3	$4^k(16n+14)$ $4n+2$	(1, 3, 3, −1, 0, 0)
8		3, 7	5	$4^k(16n+14)$ $8n+6$ $4n+1$	(2, 2, 3, −1, −1, 0)
12	1	1, 5	1, 5	$3^{2k+1}(3n+2)$ $4n+3$	(1, 1, 12, 0, 0, 0)
12	1	$s_1 = 1$		$4^k(8n+5)$	(1, 2, 6, 0, 0, 0)
12	1		3, 7	$3^{2k+1}(3n+2)$ $4n+2$	(1, 3, 4, 0, 0, 0)
12	−1	1, 5		$3^{2k+1}(3n+2)$ $4n+3$ $4n+2$	*(1, 4, 4, −2, 0, 0)
12	−1	$s_1 = -1$		$3^{2k+1}(3n+2)$ $8n+1$	(2, 2, 3, 0, 0, 0)
12	−1	3, 7	1, 5	$4^k(8n+5)$ $8n+1$	(2, 3, 3, 1, 1, 1)

f Improperly Primitive.

H	$(F \mid p)$ or $(f \mid p)$	$F \equiv$ (mod 8)	P	Forms
4		3	$4^k(8n+7)$	(2, 2, 2, 1, 1, 1)
6	1	3, 7	$3^{2k+1}(3n+1)$	(2, 2, 2, 0, 0, −1)
12	1	7	$3^{2k+1}(3n+2)$	(2, 2, 4, −1, −1, 0)
12	−1	3	$4^k(8n+5)$	(2, 2, 4, 0, 0, −1)
18	$(f \mid 3) = -1$	1, 5	$3^{2k}(3n+1)$	(2, 2, 6, 0, 0, −1)

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A DEFINITION OF GROUP BY MEANS OF THREE POSTULATES.

By Raymond Garver.

Given a set of elements $G(a, b, c, \cdots)$ and a rule of combination, which may be called multiplication, by which any two elements of G, whether they be the same or different, taken in a specified order, determine a unique result, or product, which may or may not be an element of G. This system is called a group if it satisfies certain postulates; the sets of postulates to which we shall have occasion to refer in the present paper are chosen from the following list:

I. (Closure). If a and b * are elements of G, the product ab is an element of G.

II. (Associativity). If $a, b, c, ab, bc, (ab)c, a(bc)$ are elements of G, then $(ab)c = a(bc)$.

III. (Strengthened Associativity). If $a, b, c, ab, bc, (ab)c$ are elements of G, then $(ab)c = a(bc)$.

IV. If a and b are elements of G, there exists an element x of G such that $ax = b$.

V. If a and b are elements of G, there exists an element y of G such that $ya = b$.

VI. (Existence of right-hand identity element). There exists an element e of G such that, for every element a of G, $ae = a$.

VII. (Existence of right-hand inverse element). If such elements e occur, then for a particular e and for every element a of G there exists an element a' of G such that $aa' = e$.

One important definition of group employs I, II, VI and VII. This formulation is due to Dickson (ref. 11), but it is based to a large extent on the work of Moore. These four posulates were proved by Dickson to be independent. It is worth while pointing out that postulate VII must be stated carefully; van der Waerden's form of this postulate (ref. 12) is ambiguous, as Clifford has shown (ref. 13).†

The reader is familiar with the fact that this postulate system is often met in a slightly different form, with VI and VII replaced by stronger statements which postulate identity and inverse elements, not merely *right-hand* identity and inverse elements. Dickson's work shows, of course, that it is not necessary to postulate these stronger statements.

* The symbols $a, b, \cdots$, as used in the postulates, need not represent *distinct* elements of G.

† Instead of VI and VII, van der Waerden postulates left-hand, instead of right-hand, identity and inverse elements. This is essentially the same type of definition.

The other commonly used definition of group makes use of I, II, IV and V. This set is a simplification of that used by Weber (ref. 1). He first defined a finite group by means of I, II and two other postulates whose exact form does not interest us here, and then deduced IV and V, with the uniqueness of the x and y there appearing, as theorems for finite groups. Finding that IV and V could not be so deduced for infinite groups, he added them, plus uniqueness postulates for x and y, to his set of postulates to define an infinite group. While this was a perfectly natural step to take, it led to a number of redundancies. Huntington, in 1902, showed (ref. 3) that Weber's other postulates could be deduced from I, II, IV, and V, but he did not actually emphasize having done so until 1905 (ref. 10). Moore, also in 1902, was the first to set up and study (ref. 5) the precise set I, II, IV, V.

Moore, however, left open the question as to whether I, II, IV and V form an independent set of postulates (ref. 5, page 489). I have recently been able to prove (ref. 14) that they are not independent; in fact, the closure property I can be deduced from II, IV, and V. This gives a simple definition of group by means of three postulates; further, no other postulate in the set I, II, IV and V can be deduced from the remaining three, as I have easily shown. That is, there is only one permissible three-postulate definition of group, if the three are to be chosen from I, II, IV, and V, and this set II, IV, and V is an independent set.

It should be mentioned that an earlier definition of group by three postulates was given by Huntington in 1902 (refs. 3 and 6). He employed III, IV, and V, his proof requiring the strengthened form of the associativity postulate. This definition may be thought of as the next to last step in the simplification of Weber's set of essentially 8 postulates to the set II, IV, V.

In this paper I propose to prove that a group may be defined by means of the three postulates II, IV and VI. While VI is, of course, not a weakened form of V in the sense that II is a weakened form of III, I think it will be generally agreed that VI is a "weaker" postulate than V. To justify this statement, assume that the multiplication table of the elements of G is given by means of a square array, whether finite or infinite in extent:

$$\begin{array}{c|cccc} & a & b & c & \cdots \\ \hline a & p_{11} & p_{12} & p_{13} & \cdots \\ b & p_{21} & p_{22} & p_{23} & \cdots \\ c & p_{31} & p_{32} & p_{33} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{array}$$

The products p_{ij} may or may not be elements of G. Now we see that postulate V may be thought of as a restriction on *every* column of this square array; it requires *each* column to contain *every* element of G. On the other hand, postulate VI merely restricts one column of the array; there must exist an index i such that the column of products $p_{1i}, p_{2i}, p_{3i}, \cdots$ is identical with the left border of the table $a, b, c, \cdots$. There is, I think, then some justification for the belief that the definition of group by postulates II, IV and VI is the most satisfactory, from the logical standpoint, which has yet been given.

It may be pointed out that, in the light of VI, IV may be weakened slightly by the addition of the hypothesis $a \neq b$. This is hardly an important change. It may further be noted that IV and VI may be replaced by the composite postulate

VIII. If a and b are elements of G, there exists an element x of G such that $ax = b$; if $b = a$, there exists an element e of G such that, for any a in G, we may take $x = e$.

This does not in any real sense afford a reduction to two postulates, but it does emphasize an interesting relation between IV and VI.

To prove that a group may be defined by postulates II, IV, VI, we first deduce I and then V. The deduction of V is sufficient, since, as pointed out above, I have already obtained I as a consequence of II, IV, V; I am unable, however, to obtain V without obtaining I.

Assume, then, that a and b are elements of G. We wish to show that the product ab lies in G.

(1) By VI, $\exists\, e$ in G such that $ae = a$.
(2) By VI, $ee = e$.
(3) By IV, $\exists\, c$ in G such that $ec = a$.
(4) By IV, $\exists\, d$ in G such that $ed = c$.
(5) By (2) and (4), $(ee)d = ed = c$.
(6) By (4) and (3), $e(ed) = ec = a$.
(7) By (5), (6) and II, $c = a$.
(8) By (3) and (7), $ea = a$, for any a in G.
(9) By IV, $\exists\, a'$ in G such that $aa' = e$.
(10) By IV, $\exists\, a''$ in G such that $a'a'' = e$.
(11) By (9) and (8), $(aa')a'' = ea'' = a''$.
(12) By (10) and (1), $a(a'a'') = ae = a$.
(13) By (11), (12) and II, $a'' = a$.
(14) By (13) and (10), $a'a = e$.

(15) By IV, $\exists f$ in G such that $a'f = b$.
(16) By IV, $\exists g$ in G such that $ag = f$.
(17) By (14) and (8), $(a'a)g = eg = g$.
(18) By (16) and (15), $a'(ag) = a'f = b$.
(19) By (17), (18) and II, $g = b$.
(20) By (19) and (16), $ab = f$.

Since the product $ab = f$, an element of G, property I is established. Property V then follows at once, for, if we take $y = ba'$,

(21) By II, (14) and VI, $ya = (ba')a = b(a'a) = be = b$.

We have thus exhibited an element y which satisfies V.

It is not without interest to point out that two important group properties follow easily from intermediate steps in the above proof, before closure has been deduced. Thus, at the end of step (8), we have proved that any right-hand identity element is also a left-hand identity element. Thus there exists an identity element e such that, for every element a of G, $ae = ea = a$. It then follows at once, by a familiar step, that there is a unique identity element. From (9) and (14) above we have, if a is in G, the existence of an a' in G such that $aa' = a'a = e$, in other words, the existence of an inverse element. It follows easily that, for a given a in G, the inverse a' is unique.

One question of some interest remains. If postulates II, IV and VI are sufficient to define a group, as we have showed, is the same true for the set of postulates II, V and VI? The answer is no; the simplest example of a system satisfying these postulates and yet not forming a group is given by the multiplication table

$$\begin{array}{c|cc} & e & a \\ \hline e & e & e \\ a & a & a. \end{array}$$

The set of postulates II, V and VI is related to the concept of multiple group, as defined by Clifford (ref. 13). One of the two types of multiple group, the two types not being essentially different, satisfies II, V, with uniqueness of the element y there appearing, VI, and, in addition, I. But Clifford shows that a multiple group is not, in general, a group.

Finally, postulates II, IV and VI are independent and completely independent when the number of elements in G is greater than two. (When the number of elements is two, II is a consequence of IV and VI.) Examples to prove this can be written down easily.

BIBLIOGRAPHY.

1. Weber, *Lehrbuch der Algebra*, vol. 2 (1896), pp. 3-4.
2. Pierpont, *Annals of Mathematics*, (2), vol. 2 (1900), p. 47.
3. Huntington, *Bulletin of the American Mathematical Society*, vol. 8 (1902), pp. 296-300.
4. Huntington, *ibid.*, pp. 388-391.
5. Moore, *Transactions of the American Mathematical Society*, vol. 3 (1902), pp. 485-492.
6. Huntington, *Transactions of the American Mathematical Society*, vol. 4 (1903), p. 30.
7. Moore, *Transactions of the American Mathematical Society*, vol. 5 (1904), p. 549.
8. Huntington, *Transactions of the American Mathematical Society*, vol. 6 (1905), pp. 34-35.
9. Moore, *ibid.*, pp. 179-180.
10. Huntington, *ibid.*, pp. 181-197.
11. Dickson, *ibid.*, pp. 198-204.
12. van der Waerden, *Moderne Algebra*, vol. 1 (1930), p. 15.
13. Clifford, *Annals of Mathematics*, (2), vol. 34 (1933), pp. 865-871.
14. Garver, *Bulletin of the American Mathematical Society*, vol. 40 (1934), pp. 698-701.

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THE SIMULTANEOUS REDUCTION OF TWO MATRICES TO TRIANGLE FORM.

By J. Williamson.

Introduction. A square matrix $A_0 = (a_{ij})$, $(i, j = 1, 2, \cdots, n)$, whose elements a_{ij} are complex numbers is said to be a *triangle-matrix*, if $a_{ij} = 0$, when $i > j$, or, in other words, if each element to the left of the leading diagonal is zero. The elements a_{ii}, $(i = 1, 2, \cdots, n)$, of the leading diagonal of a triangle-matrix A_0 are the latent roots or characteristic numbers of A_0. Since the sum, the difference and the product of any two triangle-matrices are all triangle-matrices, if $f(A_0, B_0) = C_0$ is a matrix polynomial in the two matrices A_0 and B_0, C_0 is a triangle-matrix. In particular, if

$$
\begin{aligned}
& C_0 = (c_{ij}), && (i, j = 1, 2, \cdots, n), \\
(1) \qquad & c_{ii} = f(a_{ii}, b_{ii}), && (i = 1, 2, \cdots, n),
\end{aligned}
$$

so that the latent root c_{ii} of C_0 is the same function of the latent roots a_{ii} and b_{ii}, that C_0 is of A_0 and B_0. Moreover, if A and B are similar to A_0 and B_0 respectively, so that there exists a non-singular matrix X satisfying the two equations

$$XA_0X^{-1} = A \text{ and } XB_0X^{-1} = B,$$

then

$$Xf(A_0, B_0)X^{-1} = f(A, B) = C,$$

and the latent roots of A, B, C are the latent roots of A_0, B_0, C_0 respectively. Consequently equation (1) is true when a_{ii}, b_{ii} and c_{ii} are the latent roots respectively of A, B and C.

Now, if $D_0 = \phi(A_0, B_0)$ is a second polynomial in the matrices A_0 and B_0, and if, when x and y are indeterminates, $\phi(x, y) \equiv f(x, y)$, $C_0 - D_0$ is a triangle-matrix whose leading diagonal is zero. For, the element in the i-th place of the leading diagonal of this matrix is,

$$c_{ii} - d_{ii} = f(a_{ii}, b_{ii}) - \phi(a_{ii}, b_{ii}) = 0.$$

Consequently $(C_0 - D_0)^n = 0$; that is, the matrix $C_0 - D_0$ is nilpotent. Hence, *if it is possible to reduce the two matrices A and B to triangle form by the same similarity transformation the matrix $f(A, B) - \phi(A, B)$ is nilpotent for every pair of polynomials f and ϕ, which satisfy the identity $f(x, y) \equiv \phi(x, y)$.*

In what follows we shall be interested in the converse of this last statement. In particular we shall show that, if certain restrictions are placed on the matrix A, a sufficient condition that it be possible to reduce A and B to triangle form by the same unitary transformation is that a finite number of matrices, each of the form $h(A)(AB-BA)$, where $h(A)$ is a polynomial in A, be nilpotent.

We shall have occasion to write an n-rowed square matrix S as a matrix of matrices,

$$S=(S_{ij}), \qquad (i,j=1,2,\cdots,t), \tag{2}$$

where S_{ij} is a matrix of e_i rows and e_j columns. If T is a second n-rowed square matrix and

$$T=(T_{ij}), \qquad (i,j=1,2,\cdots,t), \tag{3}$$

where T_{ij} is a matrix of e_i rows and e_j columns, we shall say that S and T are *similarly partitioned* or that (3) is a partition of T similar to (2). If in (2) $S_{ij}=0$, when $i\neq j$, we shall call S a *diagonal block* matrix and write

$$S=[S_1,S_2,\cdots,S_t], \tag{4}$$

where $S_i=S_{ii}$, $i=1,2,\cdots,t$.

We shall use E to denote the unit matrix and U to denote the *auxiliary unit* matrix, whose only non-zero elements lie in the diagonal above the leading one, each of which is unity.*

1. Let A be a square matrix of order n over the field of all complex numbers and let the elementary divisors of $A-\lambda E$ be

$$(\lambda-\lambda_1)^{e_1},\ (\lambda-\lambda_2)^{e_2},\cdots,\ (\lambda-\lambda_t)^{e_t},$$

where $e_i \geqq 1$ and $e_1+e_2+\cdots+e_t=n$. The classical canonical form of A is the diagonal block matrix,†

$$A_n=[M_1,M_2,\cdots,M_t]. \tag{5}$$

In (5) M_i is a square matrix of order e_i; in fact

$$M_i=\lambda_iE_i+U_i, \tag{6}$$

where E_i is the unit matrix of order e_i and U_i the auxiliary unit matrix of the same order. The matrix

$$h(A_n)=(A_n-\lambda_1E)^{r_1}(A_n-\lambda_2E)^{r_2}\cdots(A_n-\lambda_tE)^{r_t}, \qquad r_i\geqq 0, \tag{7}$$

* Cf. Turnbull and Aitken, *Canonical Matrices*, p. 62.

† Dickson, *Modern Algebraic Theories*, p. 106.

is a polynomial in A_n and is a diagonal block matrix $[N_1, N_2, \cdots, N_t]$, where

$$(8) \qquad N_i = v_{ir_i} U_i^{r_i}, \qquad (i = 1, 2, \cdots, t),$$

and

$$(9) \qquad \begin{aligned} &v_{ir_i} = 0, \quad r_i \geqq e_i, \\ &v_{ir_i} = (\lambda_i - \lambda_1)^{r_1} (\lambda_i - \lambda_2)^{r_2} \\ &\qquad \cdots (\lambda_i - \lambda_{i-1})^{r_{i-1}} (\lambda_i - \lambda_{i+1})^{r_{i+1}} \cdots (\lambda_i - \lambda_t)^{r_t}, \; r_i < e_i. \end{aligned}$$

In (7) it is understood that, if $r_i = 0$, the factor $(A_n - \lambda_i E)^{r_i}$ is replaced by the identity matrix. Let

$$(10) \qquad B_n = (B_{ij}), \qquad (i, j = 1, 2, \cdots, t),$$

be a partition of the matrix B_n similar to that of A_n in (5). Then, if $A_n B_n - B_n A_n = C$ and $C = (C_{ij})$ is a partition of C, similar to that of B_n in (10),

$$\begin{aligned} C_{ij} &= M_i B_{ij} - B_{ij} M_j \\ &= (\lambda_i E_i + U_i) B_{ij} - B_{ij} (\lambda_j E_j + U_j) \end{aligned}$$

or

$$(11) \qquad C_{ij} = (\lambda_i - \lambda_j) B_{ij} + U_i B_{ij} - B_{ij} U_j, \qquad (i, j = 1, 2, \cdots, t).$$

We shall find it convenient to use the notation $b(i, j; r, s)$ for the element in the r-th row and s-th column of the matrix B_{ij} and more generally $f(i, j; r, s)$ for the element in the r-th row and s-th column of the matrix F_{ij}, where $F = (F_{ij})$ is a partition of a matrix F similar to that of B_n in (10). With this notation equation (11) becomes

$$(12) \qquad \begin{aligned} &c(i, j; r, s) = (\lambda_i - \lambda_j) b(i, j; r, s) + b(i, j; r+1, s) - b(i, j; r, s-1), \\ &\qquad (i, j = 1, 2, \cdots, t; \; r = 1, 2, \cdots, e_i; \; s = 1, 2, \cdots, e_j), \end{aligned}$$

with the understanding that $b(i, j; e_i + 1, s) = b(i, j; r, 0) = 0$.

We now make two hypotheses;

(a) *The matrix A is not derogatory*; that is the minimum equation satisfied by A is of degree n;

(b) *For every polynomial $h(A_n)$ defined by* (7), *where $r_i = 0, 1, 2, \cdots, e_i$ and $r_1 + r_2 + \cdots + r_t \leqq n - 2$, the matrix $h(A_n)(A_n B_n - B_n A_n)$ is nilpotent.*

As a consequence of hypothesis (a) we see that the latent root λ_i of A_n is distinct from the latent root λ_j, if $i \neq j$, and accordingly that each v_{ir_i} in (9) is different from zero, when r_i is less than e_i.

Now, if

$$h(A_n)(A_n B_n - B_n A_n) = h(A_n) C = F$$

and, if $F = (F_{ij})$ is a partition of F similar to that of B_n in (10),

(13) $$F_{ij} = N_i C_{ij} = v_{ir_i} U_i^{r_i} C_{ij}, \qquad (i, j = 1, 2, \cdots, t)$$

and

(14) $$f(i, j; r, s) = v_{ir_i} c(i, j; r + r_i, s),$$
$$(i, j = 1, 2, \cdots, t;\ r = 1, 2, \cdots, e_i;\ s = 1, 2, \cdots, e_j),$$

where $c(i, j; r + r_i, s) = 0$, if $r + r_i > e_i$.

We shall now prove

LEMMA I. *If $i_1, i_2, \cdots, i_p$, is a subsequence of the sequence $1, 2, \cdots, t$ and $p \geqq 2$, then*

(15) $$b(i_1, i_2; s_1, 1) b(i_2, i_3; s_2, 1) \cdots b(i_p, i_1; s_p, 1) = 0,$$

for all positive integers $s_j \leqq e_j$.

To prove this lemma we first show that

(16) $$c(i_1, i_2; s_1, 1) c(i_2, i_3; s_2, 1) \cdots c(i_p, i_1; s_p, 1) = 0,$$

for all values of $p = 1, 2, \cdots, t$. We shall prove (16) by induction, assuming it true for $p = 1, 2, \cdots, h - 1$ and to simplify our notation shall write m_j for e_{i_j}.

If (16) is not true when $p = h$, for some set of integers $q_j \leqq m_j$ the product

(17) $$g = c(i_1, i_2; q_1, 1) c(i_2, i_3; q_2, 1) \cdots c(i_h, i_1; q_h, 1)$$

is different from zero. Moreover, if α is a positive integer and $c(i_j, i_{j+1}, q_j + \alpha, 1)$ is different from zero, q_j may be replaced by $q_j + \alpha$ in g, and the resulting product will still be different from zero. Hence we may so choose the integers q_j in (17), that

(18) $$c(i_j, i_{j+1}, q_j + \alpha, 1) = 0, \qquad \alpha > 0,$$

(19) $$c(i_j, i_{j+1}; q_j, 1) \neq 0, j = 1, 2, \cdots, h;\ i_{h+1} = i_1.$$

If $k \not\equiv j + 1 \bmod h$ and $s_j \leqq m_j$, one of the products

$$c(i_j, i_k; s_j, 1) c(i_k, i_{k+1}; q_k, 1) \cdots c(i_{j-1}, i_j; q_{j-1}, 1)$$

or

$$c(i_j, i_k; s_j, 1) c(i_k, i_{k+1}; q_k, 1) \cdots c(i_h, i_1; q_h, 1) \cdots c(i_{j-1}, i_j; q_{j-1}, 1)$$

is zero by our induction assumption, since it is of type (16) with $p \leqq h - 1$. Therefore by (19)

$$(20)\qquad c(i_j, i_k; s_j, 1) = 0,$$
$$(j, k = 1, 2, \cdots, h;\ k \not\equiv j + 1 \bmod h,\ s_j = 1, 2, \cdots, m_j).$$

Let $h(A_n)$ be the polynomial defined by (7) for which $r_k = e_k$, if k does not lie in the set $i_1, i_2, \cdots, i_h$, and $r_k = q_j - 1$, if $k = i_j$. Then, if we write v_j for $v_{iq_{j-1}}$, it follows from (9) and hypothesis (a) that $v_1, v_2, \cdots, v_h$ are all different from zero while all other v_{ik} in $h(A_n)$ are zero. Hence by (13), for this particular polynomial $h(A_n)$,

$$(21)\quad F_{ij} = 0,\ j = 1, 2, \cdots, t;\ i \text{ not in the set } i_1, i_2, \cdots, i_h,$$

and by (14) and (20),

$$(22)\quad f(i_j, i_k; s_j, 1) = v_j c(i_j, i_k; s_j + q_j - 1, 1) = 0,\ k \not\equiv j + 1 \bmod h,$$

while by (14), (18) and (19)

$$(23)\qquad \begin{cases} f(i_j, i_{j+1}; 1, 1) = v_j c(i_j, i_{j+1}; q_j, 1) \neq 0 \\ f(i_j, i_{j+1}; s_j, 1) = v_j c(i_j, i_{j+1}, s_j + q_j - 1, 1) \\ \qquad = 0,\ j = 1, 2, \cdots, h;\ i_{h+1} = i_1,\ s_j \geqq 2. \end{cases}$$

Since by hypothesis (b) the matrix F is nilpotent, so is the matrix $H = F^h$. If $H = (H_{ij})$ is a partition of H similar to that of F,

$$(24)\qquad H_{ij} = F_{i\alpha_2} F_{\alpha_2\alpha_3} \cdots F_{\alpha_h j},$$

where each α_i is summed from 1 to t. It follows from (21), that each α_i need only be summed over the set $i_1, i_2, \cdots, i_h$ and that H_{ij} is zero, if i does not lie in the set $i_1, i_2, \cdots, i_h$. Consequently H is nilpotent, if, and only, if the matrix

$$(25)\qquad Q = (H_{i_j i_k}), \qquad (j, k = 1, 2, \cdots, h),$$

is nilpotent. Moreover every matrix in the product $F_{i_j\alpha_2} F_{\alpha_2\alpha_3} \cdots F_{\alpha_h i_k}$, as a consequence of (22) and (23), is a matrix, whose first column is zero, except, perhaps, for the element in the first row. The same is therefore true of the product matrix and the element in the first row and column of this matrix is,

$$W = f(i_j, \alpha_2; 1, 1) f(\alpha_2, \alpha_3; 1, 1) \cdots f(\alpha_h, i_k; 1, 1).$$

But, by (22) and (23), W is different from zero if, and only if, $i_j = i_k$ and $\alpha_s = i_{j+s-1}$. Hence every element in the first column of $H_{i_j i_k}$ is zero, if $k \neq j$, and, consequently, every element in the first column of Q, defined by (25), except the element in the first row, is zero. The element in the first row and first column of Q is,

$$h(i_1, i_1; 1, 1) = v_1 v_2 \cdots v_h g$$

by (23) and (14). Since Q is nilpotent, $v_1v_2 \cdots v_hg = 0$ and, as $v_1v_2 \cdots v_h$ is not zero, g must be zero.

This contradiction shows that, if (16) is true when $p = h - 1$, it is also true when $p = h$. A repetition of the above argument with $h = 1$ and H replaced by F shows that (16) is true when $h = 1$, so that our proof by induction is complete and (16) is true for all values of $p \leqq t$.

In proving (16) by induction from $h - 1$ to h we use certain polynomials $h(A_n)$. Of the exponents r_i in these polynomials $h(A_n)$ only $t - h$ have their maximum value e_i, so that, if $h \geqq 2$, the sum $r_1 + r_2 + \cdots + r_t$ is at most $n - 2$. In the proof for $h = 1$, every r_i except one has its maximum value e_i; but, since,

$$c(i_1, i_1; m_1, 1) = b(i_1, i_1; m_1 + 1, 1) - b(i_1, i_1; m_1, 0) \text{ by (12)},$$
$$= 0 \text{ by definition,}$$

we do not require to use the polynomial $h(A_n)$ for which $r_{i_1} = m_1 - 1$. Hence in proving (16) we only use the $(e_1 + 1)(e_2 + 1) \cdots (e_t + 1) - (t + 1)$ polynomials $h(A_n)$ of hypothesis (b).

If $p \geqq 2$, in (16) every equation is of the type,

$$c(j, k; s_j, 1)\sigma = 0, \qquad (s_j = 1, 2, \cdots, e_j),$$

or by (12)

$$[(\lambda_j - \lambda_k)b(j, k;\ s_j, 1) + b(j, k;\ s_j + 1, 1)]\sigma = 0. \tag{26}$$

Since $\lambda_j \neq \lambda_k$, it follows that,

$$b(j, k; q, 1)\sigma = 0, \quad \text{if} \quad b(j, k; q + 1, 1)\sigma = 0,$$

and, as $b(j, k; e_j + 1, 1) = 0$ by definition, that

$$b(j, k; s_j, 1)\sigma = 0, \qquad (s_j = 1, 2, \cdots, e_j).$$

Accordingly, if $p \geqq 2$, each letter c in (16) may be replaced by a letter b, so that (15) is true and Lemma 1 is proved.

If $p = 1$ in (16), the equation corresponding to (26) is

$$b(j, j; s_j + 1, 1) = 0, \qquad (s_j = 1, 2, \cdots, e_j),$$

so that

$$b(j, j; s_j, 1) = 0, \qquad (s_j = 2, 3, \cdots, e_j), \tag{27}$$

or every element in the first column of B_{jj}, except perhaps the first is zero.

If we now write $b(i, j)$ for the column vector, whose elements form the first column of the matrix B_{ij}, equation (15) becomes

$$(28) \qquad b(i_1, i_2)b(i_2, i_3)\cdots b(i_p, i_1) = 0, \qquad 2 \leqq p \leqq t.$$

The product on the left of (28) is a symbolic one and must be interpreted to mean (15). Consequently (28) is satisfied, if, and only if, for some value of $j \leqq p$, $b(i_j, i_{j+1}) = 0$, $i_{p+1} = i_1$.

We proceed to prove

LEMMA 2. *If the t^2 vectors $b(i,j)$, $(i, j = 1, 2, \cdots, t)$, satisfy equations (28), there exists a permutation $k_1, k_2, \cdots, k_t$ of the integers $1, 2, \cdots, t$, such that $b(k_r, k_s) = 0$, if r is greater than s.*

We shall prove this lemma by induction on t assuming that it is true for $t = 2, \cdots, m - 1$. We note that the lemma is true when $m = 2$; for from the equation $b(12)\, b(21) = 0$ it follows that either $b(12)$ or $b(21) = 0$ and that the lemma is true with $k_1 = 1$, $k_2 = 2$ or $k_1 = 2$, $k_2 = 1$.

Since the vectors $b(i,j)$, $(i, j = 1, 2, \cdots, m-1)$ satisfy (15) with $t = m - 1$, by our induction assumption there exists a permutation $j_1, j_2, \cdots, j_{m-1}$ of the integers $1, 2, \cdots, m - 1$, such that $b(j_r, j_s) = 0$, if $r > s$, $(r, s = 1, 2, \cdots, m-1)$. If we write

$$g(r,s) = b(j_r, j_s) \qquad (r, s = 1, 2, \cdots, m;\ j_m = \mathrm{m}),$$

we have

$$(29) \qquad g(r,s) = 0, \qquad r > s,\ r \neq m,$$

and (15) becomes

$$(30) \qquad g(i_1, i_2)g(i_2, i_3)\cdots g(i_p, i_1) = 0, \qquad 2 \leqq p \leqq m.$$

If m does not occur in the set $i_1, i_2, \cdots, i_p$, (30) is satisfied by virtue of (29). Further, if $m = i_1$ and $i_j > i_{j+1}$ for some value of $j = 2, \cdots, p-1$, (30) is again satisfied, so that the equations (30), which are not satisfied because of (29), are all of the type

$$(31) \quad g(m, i_2)g(i_2, i_3)\cdots g(i_p, m), \qquad i_2 < i_3 < \cdots < i_p;\ 2 \leqq p \leqq m.$$

We now denote the equations (31), in which $g(j,m)$ appears, symbolically by

$$(32) \qquad \{g(j,m)\}g(j,m) = 0, \qquad (j = 1, 2, \cdots, m-1),$$

so that, if $g(j,m) \neq 0$, $\{g(j,m)\} = 0$. In (32) $\{g(j,m)\}$ represents a set of elements, each element being a product of one or more factors $g(r,s)$ and $\{g(j,m)\} = 0$ means that each element of the set is zero. In fact $\{g(j,m)\}$ is the set whose elements are

$$g(m, i_2)g(i_2, i_3)\cdots g(i_{p-1}, i_p)g(i_p, j),\ i_2 < i_3 < \cdots < i_p < j;\ 2 \leqq p \leqq j + 1.$$

But the set of elements

$$g(m, i_2)g(i_2, i_3)\cdots g(i_{p-1}, i_p),\ i_2 < i_3 < \cdots < i_p, \qquad 2 \leqq p \leqq i_p + 1,$$

is simply the set $\{g(i_p, m)\}$. Consequently,

$$(33) \quad \{g(j, m)\} = g(m, j),\ \{g(1, m)\}g(1, j), \cdots, \{g(j-1, m)\}g(j-1, j), \quad (j = 1, 2, \cdots, m-1).$$

We shall now show that for at least one value s, $1 \leqq s \leqq m$, $g(r, s) = 0$ for $(r = 1, 2, \cdots, s-1, s+1, \cdots, m)$. If $\{g(j, m)\}$ is different from zero for all values of $j = 1, 2, \cdots, m-1$, it follows from (32) that $g(j, m) = 0$, $(j = 1, 2, \cdots, m-1)$ and that we may take $s = m$. Otherwise let $\{g(s, m)\} = 0$ but $\{g(j, m)\} \neq 0$, $j \leqq s-1$; then by (33)

$$g(m, s) = g(1, s) = \cdots g(s-1, s) = 0$$

and, as by (29) $g(r, s) = 0$, when $r > s$ and $r \neq m$,

$$g(r, s) = 0, \qquad (r = 1, 2, \cdots, s-1, s+1, \cdots, m).$$

Accordingly there exists an integer s such that

$$(34) \qquad b(j_r, j_s) = 0, \quad \text{if} \quad r \neq s.$$

By our induction assumption there exists a permutation $k_2, k_3, \cdots, k_m$ of the integers $j_1, j_2, \cdots, j_{s-1}, j_{s+1}, \cdots, j_m$, such that

$$(35) \qquad b(k_r, k_f) = 0, \qquad (r, f = 2, \cdots, m,\ r > f).$$

If $j_s = k_1$, it follows from (34) and (35) that $k_1, k_2, \cdots, k_m$ is a permutation of $1, 2, \cdots, m$ of such a nature that

$$b(k_r, k_f) = 0, \qquad (r, f = 1, 2, \cdots, m;\ r > f).$$

Accordingly our lemma is proved.

Corollary. *If $t = n$, that is, if all the latent roots of A are distinct the matrix $(b_{k_r k_s})$, $(r, s = 1, 2, \cdots, n)$, is a triangle-matrix.*

This is an immediate consequence of the fact that each vector $b(k_r, k_s)$, being of dimension one, is the element $b_{k_r k_s}$.

If $k_1, k_2, \cdots, k_t$ is the permutation of $1, 2, \cdots, t$ of Lemma 2 and

$$(36) \qquad B_{k_r k_s} = D_{rs}; \qquad (r, s = 1, 2, \cdots, t),$$

the matrix $D=(D_{rs})$ is obtained from B_n by a permutation of the rows and the same permutation of the columns of B_n. But such a transformation of B_n is a similarity transformation,* so that there exists a non-singular matrix X_n satisfying the equation

$$X_n^{-1}B_nX_n = D. \tag{37}$$

By Lemma 2 and equation (27) all the elements in the first column of D are zero except perhaps the first. Hence

$$D=\begin{pmatrix} b_1 & \beta_1 \\ 0 & B_{n-1}\end{pmatrix},$$

where β_1 is a row vector of dimension $n-1$, 0 is the zero column vector of dimension $n-1$ and B_{n-1} a square matrix of order $n-1$. Similarly

$$X_n^{-1}A_nX_n=\begin{pmatrix} a_1 & \alpha_1 \\ 0 & A_{n-1}\end{pmatrix}$$

where α_1 is the vector $(1,0,\cdots,0)$ of dimension $n-1$ and A_{n-1} is a square matrix of order $n-1$. Since

$$X_n^{-1}h(A_n)(A_nB_n-B_nA_n)X_n=\begin{pmatrix} 0 & \gamma \\ 0 & C_{n-1}\end{pmatrix},$$

where $C_{n-1}=h(A_{n-1})(A_{n-1}B_{n-1}-B_{n-1}A_{n-1})$, if $h(A_n)(A_nB_n-B_nA_n)$ is nilpotent, so is $h(A_{n-1})(A_{n-1}B_{n-1}-B_{n-1}A_{n-1})$. As a consequence of the nature of the matrix X_n in (35), A_{n-1} is still in canonical form; in fact

$$A_{n-1}=[M'_{k_1}, M_{k_2},\cdots,M_{k_t}],$$

where M'_{k_1} is the matrix of $e_{k_1}-1$ rows and columns, obtained from M_{k_1} by removing the first row and the first column. Hence the polynomials of hypothesis (b), if defined for A_{n-1} instead of A_n would be $h(A_{n-1})$, where $h(A_n)$ is one of the polynomials (7) with r_{k_1} restricted to be at most $e_{k_1}-1$. Accordingly by substituting A_{n-1} and B_{n-1} for A_n and B_n respectively and repeating our proof we show the existence of a non-singular $n-1$ rowed matrix Y, such that

$$Y^{-1}A_{n-1}Y=\begin{pmatrix} a_2 & \alpha_2 \\ 0 & A_{n-2}\end{pmatrix} \quad\text{and}\quad Y^{-1}B_{n-1}Y=\begin{pmatrix} b_2 & \beta_2 \\ 0 & B_{n-2}\end{pmatrix}, \tag{38}$$

where α_2 and β_2 are row vectors of dimension $n-2$ and A_{n-2} and B_{n-2} square matrices of order $n-2$. Moreover, if

$$X_{n-1}=X_n\begin{pmatrix} 1 & 0 \\ 0 & Y\end{pmatrix},$$

* Turnbull and Aitken, *Canonical Matrices*, p. 11.

it follows from (37) and (38) that

$$X_{n-1}^{-1}A_nX_n=\begin{pmatrix} a_1 & a_{12} & \alpha_{13} \\ 0 & a_2 & \alpha_2 \\ 0 & 0 & A_{n-2}\end{pmatrix} \text{ and } X_{n-1}^{-1}B_nX_{n-1}=\begin{pmatrix} b_1 & b_{12} & \beta_{13} \\ 0 & b_2 & \beta_2 \\ 0 & 0 & B_{n-2}\end{pmatrix},$$

where the meaning of α_{13} and β_{13} is obvious. By repeating this process exactly $n-1$ times we find a non-singular matrix X_1 satisfying the equations

$$(39) \qquad X_1^{-1}A_nX_1=A_0 \quad \text{and} \quad X_1^{-1}B_nX=B_0,$$

where A_0 and B_0 are triangle-matrices. Moreover, since X_1, in (39), is of the same type as X_n, A_0 and B_0 are derived from A_n and B_n respectively by a permutation of the rows and the same permutation of the columns. The matrix B_0 may be a triangle matrix of the most general type—that is, each element to the right of the leading diagonal may be different from zero but the matrix A_0 is not, since A_n is in canonical form. In fact in each row or column of A_0 there is at most one element, outside of the leading diagonal, which is different from zero.

Since A_n is the canonical form of A there exists a non-singular matrix Z such that, $Z^{-1}AZ=A_n$. If $Z^{-1}BZ=B_n$, then $h(A_n)(A_nB_n-B_nA_n)$ is nilpotent, if, and only if, $h(A)(AB-BA)$ is nilpotent. Moreover, if $W=ZX_1$, as a consequence of (39) we have

$$(40) \qquad W^{-1}AW=A_0 \quad \text{and} \quad W^{-1}BW=B_0.$$

Accordingly we have proved,

THEOREM I. *Let A be a square matrix of order n and let the elementary divisors of $A-\lambda E$ be*

$$(\lambda-\lambda_1)^{e_1}(\lambda-\lambda_2)^{e_2}\cdots(\lambda-\lambda_t)^{e_t}, \qquad e_1+e_2+\cdots e_t=n.$$

If A is not derogatory and if $h(A)(AB-BA)$ is nilpotent for each of the $(e_1+1)(e_2+1)\cdots(e_t+1)-t-1$ polynomials

$$h(A)=(A-\lambda_1E)^{r_1}(A-\lambda_2E)^{r_2}\cdots(A-\lambda_tE)^{r_t},$$
$$0\leqq r_i\leqq e_i,\ r_1+r_2+\cdots+r_t\leqq n-2,$$

then there exists a non-singular matrix W, satisfying (40), where B_0 is a triangle-matrix and A_0 is a triangle-matrix, derived from the classical canonical form of A by a permutation of the rows and the same permutation of the columns.

COROLLARY I. *If all the latent roots of A are distinct, a necessary and sufficient condition, that it be possible to reduce A to diagonal form and B to triangle form by the same similarity transformation, is hypothesis (b).*

For in this case the matrix A_0 is a diagonal matrix. It is interesting to compare this with the simpler but stronger condition, $AB - BA = 0$, for the possibility of a simultaneous reduction of A and B both to diagonal form.

COROLLARY II. *If A has a single elementary divisor, a necessary and sufficient condition, that it be possible to reduce A to canonical form and B to triangle form by the same similarity transformation, is hypothesis (b).*

For in this case A_0 is the same as A_n, since any permutation of the columns and the same permutation of the rows would destroy its triangle form. In this case the number of polynomials $h(A)$ of hypothesis (b) is a minimum namely $n - 1$, while in the previous case the number is a maximum, namely $2^n - n - 1$.

We now show by a simple example, that, if A is derogatory, hypothesis (b) is not sufficient to ensure the conclusion of Theorem 1.

Let $$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Any polynomial $f(A)$ is of the form

$$\begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho + \sigma \end{pmatrix}$$

and accordingly,

$$f(A)(AB - BA) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\rho \\ \rho + \sigma & 0 & 0 \end{pmatrix}.$$

Since this last matrix is nilpotent for all values of ρ and σ, hypothesis (b) is certainly satisfied. Let $W^{-1}AW = A_0$ and $W^{-1}BW = B_0$, where A_0 and B_0 are triangle-matrices. Then $W^{-1}(\lambda A + \mu B)W = \lambda A_0 + \mu B_0$ identically in λ and μ, and in particular

$$(41) \qquad | \lambda A + \mu B | \equiv | \lambda A_0 + \mu B_0 |.$$

The determinant on the left of (41) has the value μ^3 and on the right the value $(\lambda + \omega_1 \mu)\omega_2\omega_3\mu^2$ where $\omega_1, \omega_2, \omega_3$ are the three cube roots of unity. Hence (41) is not true and it is impossible to reduce A and B simultaneously to triangle form. Therefore, when A is derogatory, even if hypothesis (b) is

strengthened by replacing the finite number of polynomials $h(A)$ by all polynomials $f(A)$, it is not sufficient to ensure the simultaneous reduction of A and B to triangle form.

If A is derogatory, but for some value of λ, $A+\lambda B=C$ is not derogatory, we may apply Theorem 1 to the matrices C and B. In hypothesis (b), $h(A)$ must be replaced by $h(C)$ and the nilpotent polynomials by $h(C)(CB-BC)$. The matrix $h(C)CB$ is certainly a polynomial in A and B, say $f(A,B)$, and $h(C)BC$ a second such polynomial $\phi(A,B)$. Moreover, if x and y are indeterminates

$$f(x,y)-\phi(x,y)\equiv 0. \tag{42}$$

Hence, if for every pair of polynomials f and ϕ, which satisfy (42), $f(A,B)-\phi(A,B)$ is nilpotent, it is possible to reduce C and B, and therefore A, to triangle form by the same similarity transformation. It seems probable that a similar result holds even when every matrix of the pencil is derogatory but as yet we have been unable to prove it.

As a consequence of Theorem 1, we have

THEOREM 2. *If A is not derogatory a necessary and sufficient condition that the latent roots of $f(A,B)$ be $f(\lambda_i,\mu_i)$, for every polynomial $f(A,B)$, where λ_i and μ_i are the latent roots of A and B respectively, is that hypothesis (b) be satisfied.**

For, if (b) is true, A and B can be reduced simultaneously to triangle form and hence the latent roots of $f(A,B)$ are $f(\lambda_i,\mu_i)$. Conversely if the latent roots of $f(A,B)$ are $f(\lambda_i,\mu_i)$, the latent roots of $h(A)(AB-BA)$ are all zero, so that $h(A)(AB-BA)$ is nilpotent and (b) is satisfied.

As a triangle-matrix is the canonical form of a matrix under unitary transformation † it is to be expected that a theorem similar to Theorem I should hold, if unitary transformations are employed instead of similarity transformations. This is in fact the case. Since the matrix W in (40) is non-singular there exists a triangle-matrix T such that $WT=U$ is a unitary matrix.‡ We have therefore from (40)

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* This problem has also been considered by G. S. Bruton, "Certain aspects of the theory of equations for a pair of matrices," and M. H. Ingraham, "A study of related pairs of square matrices." Abstracts of these papers appear in the *Bulletin of the American Mathematical Society*, vol. 38 (1932), p. 633. N. H. McCoy in his paper "Quasi-commutative matrices," *Transactions of the American Mathematical Society*, vol. 36 (April, 1934), shows that if A and B are quasi-commutative the latent roots of $f(A,B)$ are $f(\lambda_i,\mu_i)$.

† Turnbull and Aitken, *op. cit.*, p. 94.

‡ Turnbull and Aitken, *op. cit.*, p. 96. Schmidt's Theorem.

$$T^{-1}W^{-1}AWT = U^*AU = T^{-1}A_0T = T_1,$$
$$T^{-1}W^{-1}BWT = U^*BU = T^{-1}B_0T = T_2,$$

where, since the inverse of a triangle-matrix is a triangle-matrix, T_1 and T_2 are triangle-matrices.

Hence we have,

THEOREM 3. *If A is not derogatory, a necessary and sufficient condition, that it be possible to reduce A and B to triangle form, both by the same unitary transformation, is that hypothesis (b) be satisfied.*

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SINGULARITIES OF ANALYTIC VECTOR FUNCTIONS.

By Si-Ping Cheo.

1. *Preliminary considerations.* There are many methods of extending the theory of ordinary analytic functions to three dimensional space or better of constructing a theory of functions of three variables which would be analogous to the theory of ordinary analytic functions. For example, expansions in power series, conformal representation, Cauchy's method based on monogeneity, etc. are all capable of leading to various extensions of the theory of ordinary analytic functions. The theory we have in mind here is based on the generalization of the Cauchy-Riemann differential equations.

Definition of analytic vector functions. If we have three functions X, Y, Z of three real variables x, y, z which are Cartesian coördinates of a point in space, if all the partial derivatives of the first order exist and are continuous in a certain region R, and if the following conditions,

$$\begin{aligned}(1.1)\qquad \operatorname{div}\vec{\Phi} &= \operatorname{div}(X\vec{i}+Y\vec{j}+Z\vec{k}) = 0 \\ \operatorname{curl}\vec{\Phi} &= \operatorname{curl}(X\vec{i}+Y\vec{j}+Z\vec{k}) = 0\end{aligned}\qquad \begin{matrix}\vec{i},\ \vec{j},\ \vec{k}, \text{ unit vectors per-} \\ \text{pendicular to each other,}\end{matrix}$$

are satisfied in R, then we shall say the vector function, $\vec{\Phi}$ is analytic throughout R.

The above set of equations has been considered as a generalization of the set of the Cauchy-Riemann differential equations.*

By the fundamental theorems of vector calculus, we notice that from the first equation of (1.1) $\vec{\Phi}$ must be the curl of a vector function $\vec{\Psi}$ (say), and from $\operatorname{curl}\vec{\Phi} = 0$, $\vec{\Phi}$ must be the gradient of a scalar function H (say); thus we obtain the following relation:

$$(1.2)\qquad \operatorname{curl}\vec{\Psi} = \operatorname{grad} H.$$

From the above relation, we can easily see $\nabla^2 H = 0$ and $\operatorname{grad}\operatorname{div}\vec{\Psi} = \nabla^2\vec{\Psi}$, where ∇^2 denotes the Laplace Operator. In fact, we could state the following two lemmas:

* G. Y. Rainich, "Analytic functions and mathematical physics," *Bulletin of the American Mathematical Society* (October, 1931).

LEMMA 1. *A necessary and sufficient condition for a vector function,* $\vec{\Phi} = \operatorname{grad} H$, *to be analytic is that H must be harmonic.**

LEMMA 2. *A necessary and sufficient condition for a vector function,* $\vec{\Phi} = \operatorname{curl} \vec{\Psi}$ *to be analytic is*

$$\operatorname{grad} \operatorname{div} \vec{\Psi} = \nabla^2 \vec{\Psi}. \tag{1.3}$$

The above two lemmas suggest to us that we may have two ways of obtaining analytic vector functions from *harmonic functions.* The first consists simply in taking the gradient of a harmonic function; a function obtained in this way we shall call a *gradient function.* The second consists in going through the following steps:

1) Replacing x, y, z in a harmonic function $H(x, y, z)$ by $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$, respectively.
2) Integrating $H(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ along a close curve C_1 with respect to x_1, y_1, z_1, that is, taking $\int_{C_1} H(x_2 - x_1, y_2 - y_1, z_2 - z_1)\, \vec{ds_1}$ where $\vec{ds_1} = dx_1 \vec{i} + dy_1 \vec{j} + dz_1 \vec{k}$ is the curve element of C_1.
3) Taking the curl with respect to x_2, y_2, z_2 of $\int_{C_1} H\, \vec{ds_1}$, that is, taking $\operatorname{curl}_2 \int_{C_1} H(x_2 - x_1, y_2 - y_1, z_2 - z_1)\, \vec{ds_1}$.

We shall call this process the Ω-process; and the functions which are obtained by Ω-process will be called Ω-functions. Now we can state the following theorem:

THEOREM 1. *Ω-functions are always analytic.*

Without any difficulty, this theorem may be proved rigorously; and it is quite obvious from the view-point of mathematical physics.†

2. *Singularities.* An analytic vector function in three dimensional space may have isolated singular points, and it may also have isolated singular curves. The definitions of these singularities seem to be very natural, and are given as follows: A point is said to be an *isolated singular point* of a given analytic vector function, provided that this function is not analytic at

* In order to express ourselves briefly, we shall define a harmonic function in the following way: A function which possesses all continuous partial derivatives of the first and second orders and satisfies Laplace's equation will be called a harmonic function.

† See, for example, Livens, *Theory of Electricity* (1918), p. 356.

that point, but at all points in the neighborhood of this point, the function is ~~of~~ analytic. A curve is said to be an *isolated singular curve* of a given analytic vector function, provided that this function is not analytic at any of the points of the curve, but at all points in the neighborhood of the curve, the function is analytic.

An isolated singular point and an isolated singular curve will be called briefly a *singular point* and a *singular curve*, respectively.

We want to investigate now the singularities of the two kinds of analytic vector functions introduced in the preceding section.

If the harmonic function H which has been used in the formation of a gradient function has a singular point * at (a, b, c), then the gradient function will also have a singularity at that point. Furthermore, we notice that the operator gradient does not introduce any new singularity. Hence, we can state the following theorem:

THEOREM 2. *A gradient function possesses the same singularities as those of the corresponding harmonic function.*

Let us now investigate the singularities of Ω-functions. Consider the vector function,

$$\vec{\Phi} = \text{curl}_2 \int_{C_1} (1/\gamma_{21})\, \vec{ds}_1$$

where $\gamma^2_{21} = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$. It is well-known the function $1/\gamma_{21}$ is single-valued and harmonic everywhere in space except at the origin. Therefore $\vec{\Phi}$ is analytic everywhere in space except when $x_2 = x_1$, $y_2 = y_1$, $z_2 = z_1$; that is to say, $\vec{\Phi}$ is not analytic at every point of the curve C_1. It will be seen in the next section that C_1 in the singular curve of $\vec{\Phi}$. In general, if a single-valued harmonic function possesses a singular point at (a, b, c), then the corresponding Ω-function is analytic everywhere in space, except at all the points of $C_{1(a\vec{i}+b\vec{j}+c\vec{k})}$ which is obtained from C_1 by translating it through the vector, $a\vec{i} + b\vec{j} + c\vec{k}$. In fact, we could state the following theorem:

THEOREM 3. *If a single-valued harmonic function possesses n singular points at (a_1, b_1, c_1), (a_2, b_2, c_2), $\cdots$ (a_n, b_n, c_n), then the corresponding Ω-function will be defined and analytic everywhere in space except on the points of the n congruent curves $C_{1(a_1\vec{i}+b_1\vec{j}+c_1\vec{k})}$, $C_{1(a_2\vec{i}+b_2\vec{j}+c_2\vec{k})}$, $\cdots$, $C_{1(a_n\vec{i}+b_n\vec{j}+c_n\vec{k})}$, which are obtained from C_1 by translating it through the following vectors: $a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$, $a_2\vec{i} + b_2\vec{j} + c_2\vec{k}$, $\cdots$ $a_n\vec{i} + b_n\vec{j} + c_n\vec{k}$, respectively.*

* That is to say: H is harmonic everywhere in space except at (a, b, c).

The Ω-process breaks down for the points which lie on the curves, $C_{1(a_1\vec{i}+b_1\vec{j}+c_1\vec{k})}$, $C_{2(a_2\vec{i}+b_2\vec{j}+c_2\vec{k})}$, $\cdots$, $C_{1(a_n\vec{i}+b_n\vec{j}+c_n\vec{k})}$. Whether or not it is possible to assign values to the function at the points of these curves in such a way as to make the vector function analytic on these curves, the next section will tell us.

3. *Residues of analytic vector functions.* The first equation of (1.1) implies the vanishing of a surface integral,

$$\iint_S (Xl + Ym + Zn)\,d\sigma, \tag{3.1}$$

where S is a surface lying within the region R and which can be contracted to a point without going outside of R; and l, m, n are the direction cosines of the normal * to S. This can be seen by Gauss' Theorem, which states:

$$\iiint_V (\partial X/\partial x + \partial Y/\partial y + \partial Z/\partial z)\,d\tau = \iint_S (Xl + Ym + Zn)\,d\sigma,$$

V being the volume bounded by S.

In (1.1), $\operatorname{curl}\vec{\Phi} = 0$ is the condition for the vanishing of a curve integral:

$$\int_C (X dx + Y dy + Z dz), \tag{3.2}$$

where the curve C lies entirely in R, and can be contracted to a point without going outside R. In this case the proof is based on the following identity:

$$\iint_S \left\{ (\partial Z/\partial y - \partial Y/\partial z)l + (\partial X/\partial z - \partial Z/\partial x)m + (\partial Y/\partial x - \partial Z/\partial y)n \right\} d\sigma$$
$$= \int_C (X dx + Y dy + Z dz),$$

S being a surface bounded by C. The above relation is known as Stokes' Theorem.

It may be the case that we can not contract S, and C to a point without going outside R, then the surface integral (3.1) and the curve integral (3.2) may have values different from zero, say K_s and K_c, respectively. We shall call $(1/4\pi)K_s$ the surface-residue, and $(1/4\pi)K_c$ the curve-residue of the vector function, $\vec{\Phi}$, given by S and C respectively.

Suppose $\vec{\Phi}$ has an isolated singular point. This point must be considered as not belonging to R; therefore, the surface S enclosing this point can not

* We shall assume that the normal to be directed inward.

be contracted to a point without going outside R. In this case, the surface residue might be different from zero. We notice that this surface residue is independent from the surface which encloses the singular point. In fact, two surfaces which enclose the same singular point and no other singularities can be transformed, one from the other, without going outside the region in which the vector function is analytic; therefore, they will give the same residue. We shall call this value the *Surface-Residue* of that function with respect to the singular point.

What could prevent the integral (3.2) from being zero is the existence of a closed singular curve of the vector function, $\vec{\Phi}$. In case a curve links the singular curve, it can not be contracted to a point without going outside R. Two curves which can be transformed one into the other without going outside R give the same residue, regardless of sign. In particular, two curves each of which links a given singular curve once can be so transformed into each other; therefore, they give the same residue. We shall call the residue given by a curve which links *once* with a singular curve of a vector function, the *Curve-Residue* of the vector function with respect to the singular curve.

Summarizing the above considerations and using the notations of vector calculus, we can define these two kinds of residues of analytic functions as follows:

If S is a closed surface lying in the region of analyticity of a vector function $\vec{\Phi}$ but enclosing giving singularities of that function, then the surface integral $(1/4\pi)\int\int_S \underbrace{\vec{\Phi}, \vec{n}}_{\cdot} \, d\sigma$ will be called the surface residue of $\vec{\Phi}$ with respect to the singularities, where $\vec{n}$ the unit normal of $d\sigma$ directs toward the interior of S, $d\sigma$ in the element of S, and the dot $(\cdot)$ is used as a sign of scalar product.

The curve residue of $\vec{\Phi}$ with respect to its singularities will be defined as the curve integral $(1/4\pi)\int_C \vec{\Phi}\cdot\vec{ds}$, where C is a closed curve lying entirely in R and links only *once* with each of the singularities, $\vec{ds}$ in its "positive sense" denotes the element of C.

Suppose now a gradient function $\vec{\Phi}$ having a singular point $P_0(x_0,y_0,z_0)$ in a certain region R and taking the following form:

$$\operatorname{grad}\frac{K_s}{\gamma_0}, \quad K_s = \text{constant}, \quad \gamma_0^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2.$$

Then, by definition, the surface residue of the gradient function with regard to P_0 will be:

$$\frac{1}{4\pi}\int\!\!\int_S \operatorname{grad}\frac{K_s}{\gamma_0}\cdot\vec{n}\,d\sigma=\frac{1}{4\pi}\int\!\!\int_S\frac{\partial}{\partial n}\frac{K_s}{\gamma_0}\,d\sigma=K_s$$

as we have seen that it is true in the theory of potentials. In fact, we can state:

THEOREM 4. *The surface-residue of a gradient function with regard to a certain singular point in a certain region is a constant.**

Suppose that an Ω-function has a singular curve C_1 and take the following form:

$$\vec{\Phi}=\operatorname{curl}_2\int_{C_1}\frac{K_c}{\gamma_{21}}\,\vec{ds_1},$$

where K_c is a constant. According to the definition, the curve-residue of $\vec{\Phi}$ is:

$$\frac{1}{4\pi}\int_{C_2}\vec{\Phi}\cdot\vec{ds_2}=\frac{1}{4\pi}\int_{C_2}\operatorname{curl}_2\int_{C_1}\frac{K_c}{\gamma_{21}}\,\vec{ds_1}\cdot\vec{ds_2},\quad \vec{ds_2}=dx_2\vec{i}+dy_2\vec{j}+dz_2\vec{k}$$

$$=\frac{K_c}{4\pi}\int_{C_2}\int_{C_1}\operatorname{curl}_2\frac{1}{\gamma_{21}}\,\vec{ds_1}\cdot\vec{ds_2}$$

$$=\frac{K_c}{4\pi}\int_{C_2}\int_{C_1}\frac{(x_2-x_1)(dy_1dz_2-dz_1dy_2)}{\gamma^3_{21}}$$

$$+\frac{(y_2-y_1)(dz_1dx_2-dx_1dz_2)+(z_2-z_1)(dx_1dy_2-dy_1dx_2)}{\gamma^3_{21}}$$

$$=\frac{K_c}{4\pi}\int_{C_2}\int_{C_1}\frac{1}{\gamma^3_{21}}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1\\ dx_1 & dy_1 & dz_1\\ dx_2 & dy_2 & dz_2\end{vmatrix}$$

$$=M\,K_c,$$

where M, an integer,† denotes the number of times for which C_1 and C_2 are

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* A gradient function of the general form, $\vec{\Phi}=\operatorname{grad} H$ having a singular point at $P_0(x_0, y_0, z_0)$ may be developed around P_0 in a power series of the form,

$$\sum_{n=0}^{\infty}(hn/\gamma_0^{2n+1}),$$

where h_n is a homogeneous, harmonic function of n-th degree. We can verify that the residue of $\vec{\Phi}$ is h_0 which is a constant.

† *Gauss Werk*, Band V (1877), p. 605. See also Boeddicker, *Gauss'schen Theorie der Verschlingungen*, Stuttgart (1876); Urysohn, "Sur les multiplicités Cantoriennes," *Fundamenta Mathematicae*, vols. 7-8 (1925-26).

linked together. By the definition of curve-residue, C_1 and C_2 are linked together only *once*, therefore M is here equal to the unit. Hence, the curve-residue of the above Ω-function with respect to C_1 is K_c. In fact, we can state the following theorem:

THEOREM 5. *The curve-residue of Ω-function with respect to a certain singular curve in a certain region is a constant.**

We notice that the following integral:

$$\frac{1}{4\pi}\int_{C_1} \mathrm{curl}_1 \int_{C_2} \frac{K_c}{\gamma_{21}} \vec{ds_2} \cdot \vec{ds_1}$$

which is equal to

$$-\frac{K_c}{4\pi}\int_{C_1}\int_{C_2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ dx_2 & dy_2 & dz_2 \\ dx_1 & dy_1 & dz_1 \end{vmatrix}$$

is K_c also. Hence, we may state:

THEOREM 6. *The curve-residue the Ω-function,* $\mathrm{curl}_2 \int_{C_1} (1/\gamma_{21}) \vec{ds_1}$, *with respect to C_1 is identical to that of the Ω-function,* $\mathrm{curl}_1 \int_{C_2} (1/\gamma_{21}) \vec{ds_2}$, *with respect to C_2.*

There are many problems regarding analytic vector functions remaining unsolved. It would be very interesting to generalize all the considerations in the previous discussions; that is to say, to increase the number of dimensions, and to find the relationships between analytic functions and their different kinds of isolated singularities.

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* An Ω-function of the general form, $\vec{\Phi} = \mathrm{curl}_2 \int_{C_1} H \vec{ds_1}$, having a singular curve C_1 may be developed "around C_1" into power series of the form:

$$\sum_{n=0}^{\infty} \mathrm{curl}_2 \int_{C_1} (h_n/\gamma_{21}^{2n+1}) \vec{ds_1}$$

where h_n is a homogeneous, harmonic function of degree n. We can verify that the curve-residue of this Ω-function is h_0 which is a constant.

THE STRUCTURE OF A COMPACT CONNECTED GROUP.

By E. R. VAN KAMPEN.

I. In a recent paper Pontrjagin proved implicitly the following theorem:

If U is any nucleus of a compact group F, then U contains a closed invariant subgroup H of F, such that F/H is a (not necessarily connected) *Lie group.**

Applying this theorem to a sequence of nuclei of F, converging to the identity element 1 of F, we can construct a decreasing sequence of closed invariant subgroups H_n, also converging to 1, such that all factor groups $F_n = F/H_n$ are Lie groups. If $m > n$, the group $H_n/H_m = H_{nm}$ is a subgroup of $F/H_m = F_m$ and then F_n can be identified with the factor group F_m/H_{nm}.†

It can be proved very easily that F is uniquely determined by the sequence of groups F_n and the identities $F_n = F_m/H_{nm}$, $m > n$. It is even possible to construct F if a sequence of groups F_n and identities $F_n = F_m/H_{nm}$, $m > n$, is given, provided these identities satisfy an obvious transitive law.‡ However we will not need to construct a group by this method.

We consider connected groups F only. Then all groups F_n are connected also, and we can use the known structural properties of compact connected Lie groups § to find structural properties of F. By means of the relations $F_n = F_m/H_{nm}$ we establish in II relations between the structural elements of all groups F_n. Then a simple limiting process (described in III) allows to draw conclusions about F(IV). In V we make the analogous conclusions for certain finite covering groups of the groups F_n.

The results of these sections will be found in Theorems 1 and 2. The

* L. Pontrjagin, "Sur les groupes topologiques compacts," *Comptes Rendus*, vol. 198 (1934), p. 238. An explicit formulation and proof will be found in a paper by E. R. van Kampen to appear shortly in the *Annals of Mathematics*. A *nucleus* is an open set containing the identity element. Compare: E. R. van Kampen, "Locally bicompact abelian groups," *Annals of Mathematics*, vol. 36 (1935), no. 2, I, 2.

† Whenever no contradiction arises as a consequence, we do not hesitate to call simply isomorphic groups identical. This frequently leads to a considerable simplification in notation and language.

‡ Compare the paper by Pontrjagin mentioned above.

§ See E. Cartan, "La théorie des groupes finis et continus," *Mém. d. Sc. Math.*, Fasc. 42, p. 42. We suppose that the reader is acquainted with his results.

difference between the general case and the case of a Lie group is not greater than the minimum that was to be expected. Certain finite abelian groups have to be replaced by 0-dimensional compact abelian groups and certain finite direct products of (locally) simple groups by countable direct products.

In the remaining three sections we discuss the structure of the 0-dimensional groups occurring, the behavior of F as regards local connectedness, and a generalized idea of covering space naturally arising as a consequence of the relations between D, $D/B = F$ and F/A. (See Theorems 1 and 2.)

For the common part of two groups we write $A.B$. For the direct product of $A, B, \cdots$ we use the notation $[A + B + \cdots]$. The symbol (A, B) denotes the group generated by A and B. If a group A is a covering group of another group B we call the groups locally isomorphic. In that case the multiple isomorphism of A and B is such that for sufficiently small nuclei it is one-to-one and bicontinuous.

II. The compact connected Lie group F_n, $(n = 1, 2, \cdots)$, contains a number of (locally) simple invariant subgroups. The semi-simple subgroup S_n of F_n generated by all these simple groups has a finite group A_n in common with the centrum C_n of F_n. The factor group F_n/A_n is the direct product of C_n/A_n and S_n/A_n; and S_n/A_n is the direct product of simple Lie groups, each with degenerate centrum (consisting of 1 only).

Comparing F_m with $F_n = F_m/H_{nm}$, we see that H_{nm} must either contain any of the simple subgroups of F_m or meet it in at most a finite number of centrum elements. As a consequence we can find a (finite or infinite) sequence of simple Lie groups $\bar{S}^{(1)}, \bar{S}^{(2)}, \cdots$ each with degenerate centrum, and a non-decreasing sequence of integers $p_1, p_2, \cdots$ such that the simple groups occurring in S_n/A_n are simply isomorphic with $\bar{S}^{(1)}, \cdots, \bar{S}^{(p_n)}$.

Of course we may suppose that the subgroup $S_m{}^{(l)}$ of F_m corresponding to $\bar{S}^{(l)}$ has as image under the transformation defined by $F_n = F_m/H_{nm}$ the subgroup $S_n{}^{(l)}$ of F_n corresponding to the same $\bar{S}^{(l)}$. Here $S_n{}^{(l)}$ can be taken as the identity element of F_n, whenever $l > p_n$.

The image of the semi-simple subgroup S_m of F_m under the same transformation of F_m into F_n must be the corresponding subgroup S_n of F_n. For this is locally true and S_n is in F_n determined by its infinitesimal transformations.

But also the image of the centrum C_m of F_m is equal to the centrum C_n of F_n. The image C^*_n of C_m is obviously contained in C_n. The factor group of C^*_n in F_n is simply isomorphic with the factor group of $H_{nm}/(C_m \cdot H_{nm})$ in $F_m/C_m = S_m/A_m$. As any factor group of $S_m/A_m = [\bar{S}^{(1)} + \cdots + \bar{S}^{(p_m)}]$

has a degenerate centrum it follows immediately that F_n/C^*_n has a degenerate centrum, so that $C^*_n = C_n$.

Applying this reasoning to S_m and its centrum A_m instead of F_m and its centrum C_m we find that also A_n is the image of A_m under the transformation determined by $F_n = F_m/H_{nm}$.

III. Any invariant subgroup G_n of F_n determines uniquely a largest invariant subgroup G'_n of F, such that G'_n is transformed into G_n under the transformation determined by $F_n = F/H_n$. Suppose G_n is defined for all n; then the common part G of all groups G'_n is a well defined closed invariant subgroup of F. Suppose the image of G_m under the tranformation defined by $F_n = F_m/H_{nm}$ is contained in G_n; then G'_n decreases with increasing n, and the image of G under the transformation defined by $F_n = F/H_n$ is contained in G_n.

Finally, suppose that the image in F_n of the group G_m in F_m is equal to G_n. Then the image in F_n of the subgroup G'_m of F, under the transformation defined by $F_n = F/H_n$ $(m > n)$ is equal to G_n, so the image of G in F_n is also G_n. But then $G/(H_n \cdot G) = G_n$ and $H_n \cdot G$ is arbitrarily small in G, so that G is approximated by the groups G_n, in the way described in I for F and F_n.

IV. We apply this on the system of subgroups defined in II, finding invariant subgroups $S^{(l)}$, S, C, A of F, corresponding to the subgroups $S_n^{(l)}$, S_n, C_n, A_n of F_n.

The groups $S_n^{(l)}$ are for $p_n > l$, locally isomorphic with $\bar{S}^{(l)}$, so there are only a finite number of possibilities for the structure of $S_n^{(l)}$ and for sufficiently large n all groups $S_n^{(l)}$ (l fixed), must be simply isomorphic. But then they are simply isomorphic with $S^{(l)}$. So $S^{(l)}$ is a compact simple Lie group, locally isomorphic with $\bar{S}^{(l)}$.

It is clear that S is contained in the group generated by $S^{(1)}, \cdots, S^{(p_n)}$ and H_n and on the other hand that S contains all $S^{(l)}$. So S is equal to the group generated by all $S^{(l)}$.

We can see directly that the common part of all groups C'_n is the centrum of F, so C is the centrum of F. Applying this on S we see that A is the centrum of S.

As each image A_n of A is finite, A must be 0-dimensional; as A_n is the common part of S_n and C_n, A is the common part of S and C; as S_n and C_n together generate F_n, so S and C generate F.

Under the transformation defined by $F_n = F_m/H_{nm}$ the image of any co-set of A_m is a co-set of A_n, so there is an invariant subgroup of F_m/A_m of

which the factor group is F_n/A_n. It can be verified immediately that F_n/A_n can be obtained from F/A by taking the factor group of $(H_n, A)/A$. As F_n/A_n is the direct product of C_n/A_n and $\bar{S}^{(1)}, \cdots, \bar{S}^{(p_n)}$ we can also obtain F_n/A_n as the factor group of an arbitrarily small invariant subgroup of the direct product of C/A and all $\bar{S}^{(l)}$. So because any compact group is uniquely determined by its approximating groups, F/A must be the direct product of C/A and all groups $\bar{S}^{(l)}$.

All these results can be combined in the following theorem:

THEOREM 1. *Suppose F is a compact connected group, $S^{(l)}$, $l = 1, 2, \cdots$, are all the (locally) simple (compact) Lie groups invariant in F, S is the group generated by all $S^{(l)}$ and C is the centrum of F. Then S and C generate F, and have in common the 0-dimensional centrum A of S. The factor group F/A is simply isomorphic with the direct product of C/A and all groups $\bar{S}^{(l)}$, where $\bar{S}^{(l)}$ is the simple group with degenerate centrum locally isomorphic with $S^{(l)}$.*

COROLLARY. *If a compact connected group F has a degenerate centrum, then it is the direct product of a collection of simple Lie groups.*

V. For each group F_n we define a covering group D_n in the following way: An element of D_n is an oriented arc α in F_n beginning in the identity element of F_n. Two such elements α and β are called equal if they have the same endpoint and the simple closed curve $\alpha\beta^{-1}$ is isotopic with a curve in the maximal connected subgroup K_n of the centrum C_n of F_n. The product $\alpha\beta$ is defined as the arc $\alpha\beta'$, where β' is obtained from β by left multiplication with the endpoint of α.

As A_n is a finite group, D_n can also be defined as covering group of F_n/A_n; the simple closed curves of F_n/A_n, corresponding to the identity element of D_n are then isotopic with curves in C_n/A_n, but not with arbitrary such curves. Anyway we can see that D_n is the direct product of simply connected simple groups $\bar{\bar{S}}^{(1)}, \cdots, \bar{\bar{S}}^{(p_n)}$ (locally isomorphic with $S^{(1)}, \cdots, S^{(p_n)}$) and a group L_n, locally isomorphic with C_n/A_n (or with K_n). As simple closed curves in K_n correspond to the identity element of D_n it follows now that K_n and L_n are simply isomorphic and that D_n is compact. So D_n is a finite covering group of F_n, and it must have a finite centrum subgroup B_n, such that $D_n/B_n = F_n$. The groups B_n and L_n can only have the identity element in common.

The transformation of F_m into F_n defined by $F_n = F_m/H_{nm}$ can be used to define a transformation of D_m into D_n: As image in D_n of an element α

of D_m, we take the image in F_n of the arc α. The transformation so defined is independent of the particular arc chosen to determine the element of D_m, for the image of a simple closed curve in F_m, isotopic with a curve in K_m is a simple closed curve in F_n, isotopic with a curve in K_n. As apparently the image of a product is equal to the product of the images the transformation is a multiple isomorphism. So D_m has a certain invariant subgroup D_{nm}, such that $D_n = D_m/D_{nm}$ and that the resulting transformation of D_m into D_n is the one we are considering.

We can find a nucleus U of D_m, for which the transformation into F_m defined by $D_m/B_m = F_m$ is a homeomorphism and such that the same is true for the image V of U in D_n. Then the transformation of U into V defined by $D_n = D_m/D_{nm}$ is the same as the transformation of the corresponding nuclei in F_m and F_n defined by $F_n = F_m/H_{nm}$. It follows immediately that the transformation of D_m into D_n has the following properties:

1. The subgroup of D_m corresponding to $\bar{\bar{S}}^{(l)}$ is transformed into the subgroup of D_n corresponding to $\bar{\bar{S}}^{(l)}$. If $l > p_n$ the last subgroup is the identity element of D_n. If $l \leq p_n$ the correspondence between these two groups is a simple isomorphism.

2. The transformation of the subgroup L_m of D_m into the subgroup L_n of D_n can be obtained by applying in succession the simple isomorphism of L_m and K_m, the transformation of K_m into K_n defined by $F_n = F_m/H_{nm}$ and the simple isomorphism of K_n and L_n.

From 1 and 2 it follows that the groups D_n can be considered as approximating groups for a group D defined as the direct product of all groups $\bar{\bar{S}}^{(l)}$ and a group L simply isomorphic with the maximal connected subgroup K of the centrum C of F.

The image in D_n of the subgroup B_m of D_m is continued in B_n. For an element of B_m is a simple closed curve α in F_m; the image of α is a simple closed curve in F_n, that means an element of B_n.

So according to III the groups B_n determine an invariant subgroup B of D. As the image of B in D_n is part of B_n it is finite, so B is 0-dimensional. As L_n and the image of B in D_n have only the identity element in common, so L and B have only the identity element in common.

If B'_n is the subgroup of D corresponding to the subgroup B_n of D_n (compare III for G'_n in F corresponding to G_n in F_n), then the factor group of B'_n/B in D/B is simply isomorphic with D/B'_n. But D/B'_n is simply isomorphic with $D_n/B_n = F_n$. At the same time B'_n/B is arbitrarily small in D/B because B is the common part of all B'_n. So the group D/B is

approximated (in the sense of I) by the sequence of groups F_n. As any compact group is uniquely determined by its approximating sequence, it follows that $D/B = F$. So we have proved:

THEOREM 2. *Suppose for the group F of Theorem 1, K is the maximal connected subgroup of the centrum, $\bar{\bar{S}}^{(l)}$ is the simply connected group locally isomorphic with $S^{(l)}$ and D is the direct product of all $\bar{\bar{S}}^{(l)}$ and a group L simply isomorphic with K. Then D has a 0-dimensional invariant subgroup B meeting L only in the identity element and such that $F = D/B$. The subgroup B is uniquely determined up to automorphisms of D.*

Remark: The image of B_m in D_n is in general not equal to B_n (as might be expected after the considerations in III) but only contained in B_n. The reason is that it may be impossible to obtain D_n from D and F_n from $F = D/B$ using *one* invariant subgroup of D. Once the construction of D is completed, we can easily find a new sequence of groups approximating F and such that the image of the group corresponding to B_m is equal to the group corresponding to B_n. We have to find invariant subgroups T_n of D such that $D/T_n = D_n$ and then use the invariant subgroups $(T_n, B)/B$ to define the new factorgroups of F.

V. An investigation of the character of the two 0-dimensional abelian groups A and B shows that while B is the most general type, A is of very simple structure: A direct sum of finite cyclic groups.

The centrum of D is the direct product of the connected abelian group L and the centrum M of the direct product of all groups $\bar{\bar{S}}^{(l)}$. Investigations of Cartan * show that M is an arbitrary (compact) direct product of finite cyclic groups. As each co-set of L in the centrum of D has with B at most one element in common and has with M exactly one element in common, it follows that B is simply isomorphic with an arbitrary closed subgroup of M. As arbitrary closed subgroup of an arbitrary compact direct product of finite cyclic groups, B is an arbitrary 0-dimensional abelian group.†

On the other hand, A is the centrum of S and S is simply isomorphic

* See E. Cartan, *loc. cit.*, p. 41.

† The theorems on 0-dimensional abelian groups here used are readily verified by reducing them to corresponding theorems for their character groups. See L. Pontrjagin, *Annals of Mathematics*, vol. 35 (1934), pp. 361-388 and E. R. van Kampen, *Annals of Mathematics*, vol. 36 (1935), no. 2. The character group of $B(A)$ is an arbitrary *factor group* (*subgroup*) of a *discrete countable* direct product of finite cyclic groups. And it can be verified immediately that the character group of B is an arbitrary countable abelian group without elements of infinite order, while the character group of A is a discrete countable direct product of finite cyclic groups.

with the factor group of an arbitrary subgroup of M in the direct product of all $\bar{\bar{S}}^{(i)}$. So A is the factor group of an arbitrary closed subgroup of M. As such it is itself a (compact) direct product of finite cyclic groups.*

VI. The direct sum of all groups $\bar{\bar{S}}^{(i)}$ is locally connected, so its image S is also locally connected. If K is also locally connected, then D and its image F are locally connected. On the other hand, if F is locally connected, then its image F/A is also locally connected and so C/A is locally connected. So it is to be expected that F and some group connected with its centrum will be locally connected or not locally connected at the same time. It is quite easy to verify that F can be locally connected, while K is not locally connected. The following theorem shows the precise relationship:

THEOREM 3. *A compact connected group F is locally connected if and only if the group C/A* (defined in Theorem 1) *is locally connected.*

We only have to prove: If F is not locally connected, then F/A is not locally connected. For the local connectedness of C/A implies the local connectedness of $F/A = [C/A + S/A]$ and this will then imply the local connectedness of F.

So let us suppose that F is not locally connected. Then we can find a nucleus U of F, such that certain points of U arbitrarily near to 1 are not with 1 on a connected subset of U^2. As S is locally connected U determines a connected nucleus V of S. As A is 0-dimensional V contains a subgroup A' of A, that is at the same time closed and open in A.† F/A' cannot be locally connected. This follows from: If two points a and b of U are separated in U^2, then their images in F/A' are separated in the image of U. Suppose $U = U_a + U_b$, where U_a and U_b contain a and b and are separated in U^2. Then their images are open and do not have a point in common, so they form a separation of the image of U between the images of a and b.

So if F is not locally connected then F/A' is not locally connected; but F/A' and F/A are locally isomorphic, so F/A is also not locally connected.

VII. The relation between S and S/A, F and F/A, D and $D/B = F$ is quite interesting. In order to have the simplest possible case we consider the relation between the direct product $P = [\bar{\bar{S}}^{(1)} + \bar{\bar{S}}^{(2)} + \cdots]$ and $Q = [\bar{S}^{(1)} + \bar{S}^{(2)} + \cdots]$. Then $Q = P/M$ where M is the 0-dimensional centrum of P. As direct product of connected, simply connected groups P is itself simply connected. We can make the fundamental group of Q into a topological group

* See second footnote on previous page.

† See E. R. van Kampen, *Annals of Mathematics*, vol. 36 (1935), no. 2, I, 4. The theorem goes back to L. Pontrjagin.

by combining into an arbitrary nucleus of the fundamental group all its elements isotopic with simple closed curves in an arbitrary nucleus of Q . It is then evident that the fundamental group of Q is the group M. Furthermore P can be defined as the universal covering group of Q. For any element in P corresponds to a class of isotopic arcs joining an element of Q to the identity element. A nucleus of P can now be determined as the collection of classes of arcs in Q isotopic with arcs in some nucleus U of Q.

These considerations indicate how a theory of covering spaces can be established for spaces in which arbitrarily small simple closed curves are not deformable into a point. This is quite independent of the fact that the spaces considered here are group spaces.

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THE INTERSECTION OF CHAINS ON A TOPOLOGICAL MANIFOLD.†

By William W. Flexner.

1. In previous papers, one of them in collaboration with S. Lefschetz,‡ the author has dealt with topological manifolds. A topological manifold, M_n, is a compact separable Hausdorff space (therefore metric) which has a complete set of neighborhoods each of which is a combinatorial n-cell (F. M., p. 393). The following properties are shown in F. M. and F. M. 2 to hold for M_n: 1. the invariance of the homology characters; § 2. the standard properties of the Kronecker Index of two chains on M_n whose dimensions are p and $n-p$; 3. the Poincaré duality theorem. Property 1 was proved intrinsically, i. e. without imbedding M_n in a Euclidean space of higher dimension and using the properties of the space residual to M_n. In 2, however, the imbedding space was used to prove that every non-bounding p-cycle on M_n is cut by some $(n-p)$-cycle on M_n with a Kronecker Index ± 1. From 2 follows 3.

The present article makes no use of the imbedding theorem but defines intrinsically on M_n intersection cycles Γ_h $(h = p + q - n)$, for two chains, C_p and C_q, on M_n of dimensionality p and q, not meeting one another's boundaries; and proves intrinsically that the cycles thus obtained form a locally homologous family (L. T., p. 183) about the geometric intersection, G, (L. T., p. 182) of C_p and C_q, thereby duplicating for M_n the salient theorem of the Lefschetz intersection theory for simplicial manifolds.

2. Some of the proofs to follow are complex. Therefore paragraphs 2-5 contain an outline describing without details the principal theorems and the methods used in their proof.

It is first shown that if M_n is orientable,¶ there is an orientable funda-

† Received December 15, 1934.

‡ S. Lefschetz and W. W. Flexner, *Proceedings of the National Academy of Sciences*, vol. 16 (1930), pp. 530-533; W. W. Flexner, *Annals of Mathematics*, (2), vol. 32 (1931), pp. 393-406 and pp. 539-548 (F. M., F. M. 2 in the sequel).

§ Terms and notation as in S. Lefschetz, "Colloquium lectures on topology," *American Mathematical Society Colloquium Publications*, vol. 12 (1930) (L. T. in the sequel).

¶ F. M., p. 399 *et seq.* On p. 400 the lines under the first formulas should read: "If the orientation of the cells $E_n{}^i$ can be so chosen that for all i and j and all regions R, all the ε's are positive, M_n is orientable with respect to the covering $\{E_n{}^i\}$, otherwise not."

mental cycle on M_n, and vice versa. If M_n is not orientable, the work is tacitly assumed to be carried out modulo 2. Then the construction of a typical intersection cycle, Γ_h, and the proof of the locally homologous family property are made. To define Γ_h, a covering of M_n by combinatorial n-cells, $E_n^1, \cdots, E_n^r$, is now chosen and each E_n^i oriented concordantly with M_n. An intersection cycle, Γ_h, is then built up using this covering as follows. The chain C_p is deformed into a chain A_p^1, the part of C_p not on E_n^1 being left invariant, the part on E_n^1 being deformed into a chain, C_p^1, of the complex K_n^1 on E_n^1. The deformation chain of the boundary is then added. Similarly C_q is deformed into A_q^1 except that the dual, $K^*{}_n^1$, is used instead of K_n^1. The part to be deformed is so chosen that its boundary is far from $F(E_n^1)$ ($F(A)$ means "boundary of A"). The chain $C^*{}_q^1$ is then defined as the subchain of A_q^1 on $K^*{}_n^1$. As a result $F(C_p^1) \cdot C^*{}_q^1 = 0$. The chain $C_h^1 = C_p^1 \cdot C^*{}_q^1$ is then defined as in L. T., ch. iv and it appears that $F(C_h^1) = C_p^1 \cdot F(C^*{}_q^1)$. Next the part of the intersection on E_n^2 is considered. The chain A_p^1 is deformed into A_p^2 just as C_p was into A_p^1 except that the deformation must be smaller, but A_q^1 is treated differently. Only the parts of A_q^1 on E_n^2, far from $F(E_n^2)$ and not in $C^*{}_q^1$ are deformed onto $K^*{}_n^2$; the other points are left invariant. The deformation chain is added and $C^*{}_q^2$ is defined as the chain on $K^*{}_n^2$. C_h^2 is then $C_p^2 \cdot C^*{}_q^2$ and again $F(C_h^2) = C_p^2 \cdot F(C^*{}_q^2)$. By an inductive construction, this process is kept up until all cells covering the geometric intersection have been treated, thus giving fragmentary intersections: $C_h^1, C_h^2, \cdots, C_h^r$.

3. It is now necessary to connect the boundaries of these fragments properly to make a cycle. If C^{*12}_{q-1} is the part of $F(C^*{}_q^1)$ in E_n^2, then by the Lefschetz intersection theory, for every $\varepsilon > 0$, if the deformations producing A_p^2, A_q^2 are small enough, there is a singular chain, C_h^{12}, and a subchain, D^{*12}_{q-1}, of $F(C^*{}_q^2)$ such that $C_h^{12} \to C_p^1 \cdot C^{*12}_{q-1} - C_p^2 \cdot D^{*12}_{q-1} \bmod M^{12}$ near G where M^{12} is the ε-neighborhood of the complex carrying $C_p^1 \cdot F(C^{*12}_{q-1})$. A theorem of this type is then proved for an arbitrary pair of overlapping n-cells, E_n^i, E_n^j, $i < j$, so C_h^{ij} provides a connection mod M^{ij} between parts of the boundaries of C_h^i and C_h^j. The cells of $F(C_h^i)$ not on the boundary of some C_h^{gi} or C_h^{ik}, $g < i < k$, can be shown to be in ΣM^{ab}, if the deformations are small enough. The neighborhood M^{ab}, defined analogously to M^{12} is an arbitrarily small one about the $(h-2)$-complex carrying $C_p^a \cdot F(C^{*ab}_{q-1})$, which is determined at the a-th step of the construction. So $F(\sum_{i=1}^{r} C_h^i - \sum_{i,j=1}^{r} C_h^{ij}) = Q_{h-1}$ is an $(h-1)$-chain in an arbitrarily small neighborhood of an $(h-2)$-complex, and there

is a V_h such that $V_h \to Q_{h-1}$. Therefore $\Gamma_h = \Sigma C_h{}^i - \Sigma C_h{}^{ij} - V_h$ is a cycle and may be defined as an *intersection cycle* of C_p and C_q.

4. Clearly Γ_h, as defined, is a function of C_p, C_q, the n-cells $E_n{}^i$, and their order, and the sizes and characters of the various deformations. It can, however, be proved that any two intersection cycles derived from C_p and C_q are homologous in a preassigned neighborhood of G if the deformations giving rise to them are small enough, independent of the other factors.

To show this it is first proved that if Γ_h was obtained by small enough deformations, the chains giving rise to Γ_h can be further deformed to make an intersection cycle, Λ_h, on a covering $U_n, E_n{}^1, E_n{}^2, \cdots, E_n{}^r$ where U_n is any n-cell of a covering of M_n. If the new deformations are small enough, $\Gamma_h \sim \Lambda_h$ close to G. This is the substance of Lemma 1 (No. 27) in the sequel. Repeated applications of Lemma 1 make it possible to derive from a given intersection another, homologous to it, on any other covering.

5. So it is sufficient for the general homology proof to show in addition to Lemma 1 that any two intersections, Γ_h and $\hat{\Gamma}_h$, on the same covering are homologous if they are obtained from C_p and C_q by small enough deformations. This is the substance of Lemma 2 (No. 27). The same notation is used as in the construction of Γ_h except that circumflex accents are used for quantities referring to $\hat{\Gamma}_h$. The proof will now be outlined. It is, roughly speaking,† the intersection of the final deforms $A_p{}^r$ and $A_q{}^r$ of C_p and C_q whose intersection gives Γ_h. Similarly an $\hat{A}_p{}^r$ and an $\hat{A}_q{}^r$ lead to $\hat{\Gamma}_h$. Because $A_s{}^r$ and $\hat{A}_s{}^r$ $(s = p, q)$ originate from C_s there are chains $W_{s+1} \to A_s{}^r - \hat{A}_s{}^r$. The chain W_{p+1} can be deformed piece by piece onto $K_n{}^1, K_n{}^2, \cdots$ much as C_p was, leaving $C_p{}^i$ and $\hat{C}_p{}^i$ invariant for every i. Similarly W_{q+1} is deformed step by step onto $K^*{}_n{}^1, K^*{}_n{}^2, \cdots$. If the part of W_{p+1} on $K_n{}^k$ is $C^k{}_{p+1}$ and that of W_{q+1} on $K^*{}_n{}^k$ is C^{*k}_{q+1}, then calculation of boundaries (L. T., p. 169) plus the fact that $F(C_p{}^k) \cdot C^*{}_q{}^k = 0$ and $F(\hat{C}_p{}^k) \cdot \hat{C}^*{}_q{}^k = 0$ gives that

$$(-1)^{n-q} C^k{}_{p+1} \cdot C^*{}_q{}^k + \hat{C}_p{}^k \cdot C^{*k}_{q+1},$$

called $C^k{}_{h+1}$, is bounded by $C_p{}^k \cdot C^*{}_q{}^k - \hat{C}_p{}^k \cdot \hat{C}^*{}_q{}^k + X_h{}^k$. The chain $X_h{}^k$ is a combination of chains near X where X is the corresponding combination reached at the preceding stages. This gives $C^k{}_{h+1} \to C_h{}^k - \hat{C}_h{}^k + X_h{}^k$. The simplicial parts of Γ_h and $\hat{\Gamma}_h$ are $\Sigma C_h{}^k$ and $\Sigma \hat{C}_h{}^k$, so $\Sigma C^k{}_{h+1}$ is bounded by these

† The statements that follow here are none of them exactly correct, but are made to bring out the general methods of the proof. The proof in detail is given in Nos. 31-34. In comparing the chains here with those of the same name in Nos. 31-34, it is, therefore, important to note that the correspondence is only schematic.

simplicial parts plus the X's. A study of each X_h^k in relation to its predecessors similar to that of $C_p^2 \cdot F(D^{*12}_{q-1})$ in relation to $C_p^1 \cdot F(C^{*12}_{q-1})$ shows that the X's and the non-simplicial parts of Γ_h and $\hat{\Gamma}_h$ can be used to make links between the pieces C^k_{h+1} in such a way as to give $\Gamma_h \sim \hat{\Gamma}_h$.

6. *Orientation of M_n.* Since M_n is connected, it follows, as in F. M. 2, p. 548, that there is one and only one independent non-bounding n-cycle, Γ_n, on M_n to a multiple of which every n-cycle is homologous. If M_n is orientable in the sense of F. M., p. 399, Γ_n will be oriented. Conversely, if Γ_n is an oriented cycle $\not\sim 0$ on M_n, then M_n is orientable according to F. M. with respect to any covering $E_n^1, E_n^2, \cdots, E_n^r$. This is because the part of Γ_n on each E_n^i orients that E_n^i (see L. T., p. 44 and p. 101).

7. The next paragraphs deal with the definition of an intersection cycle for two chains. Being given two oriented chains C_p and C_q on M_n, assuming M_n orientable, such that $F(C_p)$ is nowhere nearer to C_q than $\alpha > 0$, and $F(C_q)$ is nowhere nearer to C_p than α, it is desired to find a semi-simplicial cycle, Γ_h, (F. M., p. 540) of dimensionality $h = p + q - n$ arbitrarily near the geometric intersection, G, and playing the rôle of an "intersection cycle."

8. Let $E_n^1, E_n^2, \cdots, E_n^r$ be the subset covering G of a covering of M_n (F. M., p. 395). There will, by definition of a covering, be a $\beta > 0$ such that every point of G has on M_n a neighborhood around it in some E_n^i of the subset with diameter β.

9. *Fundamental construction.* Choose a $\delta > 0$. Step 1. On E_n^1 take a complex K_n^1 of mesh (L. T., p. 85) $\varepsilon_1/2 < \alpha/20r$ and $< \beta/20r$, where r is as defined in No. 8. If $K^*{}_n^1$ is the dual (L. T., p. 132) on E_n^1 of K_n^1, it is of mesh ε_1. Subdivide the chains C_p and C_q until the mesh of their cells is $\varepsilon_1/2$ and call the subdivided chains by the same names again. Next deform C_p into a chain A_p^1 by means of an $\varepsilon_1/2$-deformation, as follows. Leave unaltered the closed p-cells of C_p not entirely on K_n^1. Deform the remainder onto a subchain of K_n^1 and call the new chain on K_n^1, C_p^1. Add the deformation chain of the boundary of the piece which was deformed.

10. Deform C_q in the same manner, using $K^*{}_n^1$ instead of K_n^1, and leaving invariant all q-cells of C_q not on E_n^1 and not at a distance of more than $4\varepsilon_1$ from $F(E_n^1)$. Add the deformation chain; call the deformed chain A_q^1, and the part of A_q^1 on $K^*{}_n^1$, $C^*{}_q^1$. Let $C_h^1 = C_p^1 \cdot C^*{}_q^1$, $h = p + q - n$. If ε_1 is small enough, C_h^1 is within δ of G.

11. Since all points of $F(C_p{}^1)$ must lie within ε_1 of $F(E_n{}^1)$, or by choice of ε_1 (see No. 7) be far from C_q, no point of $C^*{}_q{}^1$ can meet $F(C_p{}^1)$. Therefore (L. T., p. 169):

THEOREM A^1. $F(C_h{}^1) = C_p{}^1 \cdot F(C^*{}_q{}^1)$.

12. Assume steps $2, 3, \cdots, k-1$ to have been taken, Theorems $A^2, \cdots, A^{k-1}$ to have been proved, and, for $i < j < k$, the following chains to have been defined: $A_p{}^j$, $A_q{}^j$, $sA_p{}^j$, $sA_q{}^j$, $C_p{}^j$, $C^*{}_q{}^j$, $C_h{}^j$, C^{*ij}_{q-1}, D^{*ij}_{q-1}, $R_q{}^i$, $\Delta_q{}^{ij}$. Let C^{*ik}_{q-1} be the chain sum of the closed $(q-1)$-cells of $F(C^*{}_q{}^i)$, $i < k$, which are entirely in $E_n{}^k$ with no point within $4\varepsilon_1$ of $F(E_n{}^k)$, and which have no interior point in C^{*ij}_{q-1} or D^{*ai}_{q-1}, $a < i < j < k$.†

13. Step k. Take on $E_n{}^k$ a complex, $K_n{}^k$, of mesh ε_k, where $6\varepsilon_k < \varepsilon_{k-1}$, and ε_k satisfies other conditions to be specified later (Nos. 15, 16 and Theorems B, C, D). Let $K^*{}_n{}^k$ be the dual of $K_n{}^k$. Subdivide the chains $A_a{}^{k-1}$, $a = p, q$, into chains, $sA_a{}^{k-1}$ of mesh $\varepsilon_k/2$.

14. Now deform $sA_p{}^{k-1}$ into $A_p{}^k$ just as, in No. 10, C_p was deformed into $A_p{}^1$, using $K_n{}^k$ instead of $K_n{}^1$, and call the part of the new chain on $K_n{}^k$, $C_p{}^k$.

15. By an ε_k-deformation carry $sA_q{}^{k-1}$ into a chain $A_q{}^k$: the deformation to be as follows. It shall carry a chain $R_q{}^{k-1}$ into a subcomplex, $C^*{}_q{}^k$, of $K^*{}_n{}^k$. The chain $R_q{}^{k-1}$ is made up of the closed q-cells of $sA_q{}^{k-1}$ in $E_n{}^k$ and at a distance of more than $4\varepsilon_1$ from $F(E_n{}^k)$, but minus the cells which are 1) in $C^*{}_q{}^i$, $i < k$; 2) in $\Delta_q{}^{ij}$, the deformation chain joining C^{*ij}_{q-1} and D^{*ij}_{q-1} $i < j < k$; plus 3) such cells of $sA_q{}^{k-1}$ in $E_n{}^k$ and not in 1) or 2) as have, for some j, a point of the subdivided C^{*jk}_{q-1} but no $(q-1)$-cell of the subdivided C^{*js}_{q-1} or D^{*sj}_{q-1}, $s < k$, on their boundaries. In other words, $R_q{}^{k-1}$ is the chain sum of the closed q-cells of $sA_q{}^{k-1}$ well inside $E_n{}^k$, with C^{*jk}_{q-1} on its boundary for every j, but no $(q-1)$-cells of C^{*js}_{q-1} or D^{*sj}_{q-1} on its boundary.

All points not in $R_q{}^{k-1}$ are left invariant. Add the deformation chain of the boundary of $R_q{}^{k-1}$. Let $C_h{}^k = C_p{}^k \cdot C^*{}_q{}^k$. Assuming that at each previous stage $C_h{}^i$ was within $i\delta$ of G, $i < k$, by taking ε_k small enough, $C_h{}^k$ may be brought within $k\delta$ of G, justifying the assumption.

16. Let D^{*ik}_{q-1}, $i < k$, be the image in $F(C^*{}_q{}^k)$, under the deformation just defined, of C^{*ik}_{q-1}. By condition 3, No. 15, this image exists. Take ε_k so small that no cell of D^{*ij}_{q-1} is within $2\varepsilon_1$ of $F(E_n{}^k)$.

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† As defined, C^{*ik}_{q-1} is the part of the boundary of $F(C^*{}_q{}^i)$ which is in $E_n{}^k$ but not in $E_n{}^j$, $j < k$.

17. All points of $F(C_p^k)$ must be within ε_k of $F(E_n^k)$ or else far from $C^*{}_q^k$, for the deformations are too small to bring images of $F(C_p)$ and C_q together. So, since $C^*{}_q^k$ is entirely farther than $2\varepsilon_1 > \varepsilon_k$ from $F(E_n^k)$, $F(C_p^k) \cdot C^*{}_q^k = 0$. Therefore (L. T., p. 169):

THEOREM A^k. $F(C_h^k) = C_p^k \cdot F(C^*{}_q^k)$.

The construction and proof given here is carried out until, at the r-th stage, all n-cells $E_n^1, E_n^2, \cdots, E_n^r$ have been treated.

18. THEOREM B. *If* $1 \leq i < j \leq r$ *and if* $\varepsilon_{i+1}, \varepsilon_{i+2}, \cdots, \varepsilon_j$ *are small enough, then* $C_p^i \cdot C^{*ij}_{q-1} \sim C_p^j \cdot D^{*ij}_{q-1}$ mod M^{ij} *on* N^{ij}, *where* M^{ij} *is a* τ^{ij}*-neighborhood of* $| C_p^i \cdot F(C^{*ij}_{q-1}) |$,† τ^{ij} *arbitrary* (*but is to be chosen* $< \delta$), *and* N^{ij} *is a* ρ^i*-neighborhood of* $| C_p^i \cdot C^{*ij}_{q-1} |$. *The values of* τ^{ij} *give a maximum value to* ρ^i, *but* ρ^i *approaches zero with* $\varepsilon_{i+1}, \varepsilon_{i+2}, \cdots, \varepsilon_j$ *independent of* τ^{ij}, *and so can be taken* $< \delta$.

19. Note that $\varepsilon_{i+1}, \varepsilon_{i+2}, \cdots, \varepsilon_j$ are determined after the i-th step of the fundamental construction (referred to in the sequel as f. c.) whereas $| C_p^i \cdot F(C^{*ij}_{q-1}) |$ was determined and fixed previously, at the i-th step.

20. *Proof of B.* Both $C_p^i \cdot C^{*ij}_{q-1}$ and $C_p^j \cdot D^{*ij}_{q-1}$ are intersections in the sense of L. T., ch. iv, of the chains C_p^i and C^{*ij}_{q-1} which do not meet one another's boundaries modulo M^{ij}. They are, therefore, homologous as stated if the ε's are small enough. Since the distance from $A_p^i - C_p^i$ to C^{*ij}_{q-1} is greater than zero and depends on the ε's, no points of A_p^i not in C_p^i can have images in A_p^j meeting D^{*ij}_{q-1} provided that the ε's are small enough.

21. Now let E_{h-1} be a closed $(h-1)$-cell of $F(C_h^k)$, $1 \leq k \leq r$, and suppose $E_{h-1} = C_p^k \cdot E^*{}_{q-1}$ where $E^*{}_{q-1}$ is a closed $(q-1)$-cell of $F(C^*{}_q^k)$. Further call E_{q-1} any one of the closed $(q-1)$-cells of sA_q^{k-1} of which $E^*{}_{q-1}$ is an image. There are point sets, $\mathcal{E}$, of which E_{q-1} is image in each A_q^a, $1 \leq a < k-1$, and because regular subdivision was used in f. c., no $\mathcal{E}$ has points in more than one closed q-cell of A_q^a. Let E_q^a be a closed q-cell of A_q^a carrying an $\mathcal{E}$.

22. THEOREM C. *If* $1 \leq k \leq r$, *all cells* E_{h-1} *which are not cells of* $C_p^k \cdot D^{*jk}_{q-1}$ *or in* M^{ij}, $i < j < k$, *are cells of some* $C_p^k \cdot C^{*ks}_{q-1}$, $k < s \leq r$, *provided* $\varepsilon_{i+1}, \varepsilon_{i+2}, \cdots, \varepsilon_k$ *are small enough.*

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† Following the recent usage of S. Lefschetz: if A is a simplicial chain, $| A |$ is the complex carrying A.

Proof. In order to show that E_{h-1} is in some $C_p^k \cdot C^{*ks}_{q-1}$ it is, because of f. c., sufficient to show that E_{q-1} is not in an $F(C^*{}_q{}^j)$, $j < k$, and that E_{h-1} is within some $E_n{}^s$ by a distance of at least $4\varepsilon_1$.

Because of the condition of No. 7 and the smallness of $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$, neither $F(C_p)$ nor $F(C_q)$ nor their deforms play any rôle in $F(C_h{}^k)$. So for E_{h-1} to be in $F(C_h{}^k)$, E_{q-1} must either be

1) a cell in $F(C^*{}_q{}^j)$, $j < k$, (now denoting chain and subdivision by $F(C^*{}_q{}^j)$);

2) a cell of $F(\Delta_q{}^{ij})$, $i < j < k$, (see No. 15) or the image of such a cell;

or 3) a cell within $4\varepsilon_1 + \varepsilon_k$ of $F(E_n{}^k)$ and not in 1) or 2).

In case 1), every cell of $F(C^*{}_q{}^j)$ either belongs to C^{*jk}_{q-1} or is not deformed onto $K^*{}_n{}^k$ (condition 3, No. 15). Therefore the images of such cells are in D^{*jk}_{q-1}.

In case 2) let $\Delta'_q{}^{ij}$ be the image of $\Delta_q{}^{ij}$ and let $\Delta^{ij}_{q-1} = F(\Delta'_q{}^{ij}) - (C^{*ij}_{q-1} - D^{*ij}_{q-1})$. Now let $\Delta^{*ij,k}_{q-1}$ be the part of Δ^{ij}_{q-1} in $F(C^*{}_q{}^k)$. If $C^k \cdot \Delta^{*ij,k}_{q-1}$ did not lie in M^{ij} when the ε's are small enough, there would be for each of an infinite number of sets of these ε's as they approached zero, a point, P, of the corresponding $C_p{}^k \cdot \Delta^{*ij,k}_{q-1}$ at a distance, $d(P) > \rho > 0$ from $| C_p{}^i \cdot F(C^{*ij}_{q-1}) |$, which complex is not a function of the ε's mentioned. The points P would then have a limit point, L, at a distance $\geqq \rho$ from $| C_p{}^i \cdot F(C^{*ij}_{q-1}) |$.

i. Suppose L is not on $| F(C^{*ij}_{q-1}) |$ but has a distance $d' > 0$ from it. The chain Δ^{ij}_{q-1} being the image of the deformation chain of $F(C^{*ij}_{q-1})$ would, if $\varepsilon_j + \varepsilon_{j+1} + \cdots + \varepsilon_k < d'/4$, lie within $d'/4$ of $| F(C^{*ij}_{q-1}) |$, so $C_p{}^k \cdot \Delta^{*ij,k}_{q-1}$, a subset, would also; and the points P would be, after a certain one, all within $d'/2$ of $| F(C^{*ij}_{q-1}) |$ contrary to the hypothesis that L is their limit point. Thus case i. cannot occur.

ii. Suppose, then, that L is on $| F(C^{*ij}_{q-1}) |$. Since by f. c., no points of $A_p{}^i$ not on $C_p{}^i$ can meet $| F(C^{*ij}_{q-1}) |$, it is possible by taking the ε's small enough to bring the point set intersection of $| C_p{}^k |$ and $| F(C^{*ij}_{q-1}) |$ arbitrarily close to $| C_p{}^i \cdot F(C^{*ij}_{q-1}) |$. Then the points P, since each is on a $C_p{}^k$ and near $| F(C^{*ij}_{q-1}) |$, will again, after a certain one, be nearer by a finite amount to $| C_p{}^i \cdot F(C^{*ij}_{q-1}) |$ than L is. So since i. and ii., are exhaustive, $C_p{}^k \cdot \Delta^{*ij,k}_{q-1}$ must be in M^{ij}.

In case 3) there must be a first n-cell, $E_n{}^a$, $a \neq k$ containing E_{q-1} in such a way that all its points are at a distance of at least $\beta/2$ from $F(E_n{}^a)$ (condition of No. 8). If $a > k$ the theorem is proved. If $a < k$ there is in $E_n{}^a$ and $E_q{}^a$ of the type defined in No. 21 which must be inside $E_n{}^a$ by a margin of $[\beta/2 - (\varepsilon_k + \varepsilon_{k-1} + \cdots + \varepsilon_a)] > 4\varepsilon_1$. Therefore $E_q{}^a$ is a cell of $F(C^*{}_q{}^j)$, $j \leqq a$, or $\Delta_q{}^{xy}$, $x < y < a$. This reduces case 3) to cases 1) and 2) already considered, since the only cells of $\Delta_q{}^{xy}$ deformed are on $F(\Delta_q{}^{xy})$.

23. THEOREM D. *If E_{h-1} is in $C_p{}^k \cdot D^{*ik}_{q-1}$ and in $C_p{}^k \cdot D^{*jk}_{q-1}$, $i < j < k$, it is in M^{ij} provided $\varepsilon_{i+1}, \varepsilon_{i+2}, \cdots, \varepsilon_k$ are small enough.*

If $E^i{}_{q-1}$ and $E^j{}_{q-1}$ are the originals of $E^*{}_{q-1}$ in C^{*ik}_{q-1} and C^{*jk}_{q-1} respectively, then they must be within ε_k of each other. Then $E^j{}_{q-1}$ must be within $2\varepsilon_j$ of $|F(C^{*ik}_{q-1})|$; for, since ε_k is less than the meshes of both $K^*{}_n{}^i$ and $K^*{}_n{}^j$, it is only by being in a q-cell of C^{*jk}_{q-1} abutting on $|F(C^{*ik}_{q-1})|$ that $E^j{}_{q-1}$ can be within ε_k of C^{*ik}_{q-1}. Therefore E_{q-1} is within $2\varepsilon_j + \varepsilon_k$ of $|F(C^{*ik}_{q-1})|$. But E_{h-1} is also on a part of $C_p{}^k$ which was obtained by an $\varepsilon_{i+1} + \varepsilon_{i+2} + \cdots + \varepsilon_k$ deformation from $C_p{}^i$. Therefore if these ε's are small enough, E_{q-1} is within τ^{ik} of $|C_p{}^i \cdot F(C^{*ik}_{q-1})|$, i. e. in M^{ik}. (Note once more that M^{ik} is independent of $\varepsilon_{i+1}, \varepsilon_{i+2}, \cdots, \varepsilon_k$).

24. THEOREM E.

$$F(C_h{}^k) = -\sum_{i=1}^{k-1} C_p{}^k \cdot D^{*ik}_{q-1} + \sum_{i=k+1}^{r} C_p{}^k \cdot C^{*ki}_{q-1} \bmod \Sigma M^{ab}.$$

It is a consequence of f. c. and Theorems C and D that each cell of $F(C_h{}^k)$ not in M^{ab} is in one and only one of the chains on the right-hand side of the formula above. It remains to make sure of the coefficients in each case. Those cells in the second term have the right coefficient by Theorem A^k and the definition of C^{*ki}_{q-1}. As to the first term, since C_q is an oriented chain, D^{*ik}_{q-1} is negatively related to $F(C^*{}_q{}^k)$, so $C_p{}^k \cdot D^{*ik}_{q-1}$ is negatively related to $F(C_h{}^k)$.

25. The condition of No. 8 makes it sure that f. c. comes to an end at the r-th step: all $(h-1)$-cells of $C_p{}^r \cdot F(C^*{}_q{}^r)$ belong either in $-C_p{}^r \cdot D^{*jr}_{q-1}$, $j < r$, or in $M = \Sigma M^{ab}$ since there can be no C^{*rs}_{q-1}, $s > r$.

Form the sum

$$C_h = \sum_{k=1}^{r} C_h{}^k - \sum_{i,k=1}^{r(i<k)} C_h{}^{ik}$$

where $$C_h{}^{ik} \rightarrow C_p{}^i \cdot C^{*ik}_{q-1} - C_p{}^k \cdot D^{*ik}_{q-1} \bmod M^{ik} \text{ on } N^{ik}.$$

The existence of $C_h{}^{ik}$ follows from Theorem B. Since it is within $(r+1)\delta$ of G, C_h is arbitrarily close to G. Computing the boundary of C_h formally (L. T., p. 169) and using Theorems B and E and the Theorems A, gives that $F(C_h)$ is an $(h-1)$-cycle on M. But M is an arbitrarily small neighborhood of an $(h-2)$-complex, so, as in F. M. 2, p. 541, $F(C_h) \sim 0$ on M' where M' is a neighborhood of M whose size approaches zero with the size of M, and whose distance from G approaches zero with the size of M. Therefore there is a complex, V_h, on M' such that $V_h \rightarrow F(C_h)$.

26. Then $\Gamma_h = C_h - V_h$ is a semi-simplicial h-cycle on M_n arbitrarily close to G. The cycle is defined as an *intersection cycle on the covering* $E_n^1, E_n^2, \cdots, E_n^r$ of the chains C_p and C_q.

27. The next numbers will be devoted to the proof of the following theorem.

THEOREM F. *Two intersection cycles Γ_h and $\hat{\Gamma}_h$ of the chains C_p and C_q are homologous in any arbitrarily small given neighborhood of G provided the deformations used in getting them are small enough, even if Γ_h is on the covering $E_n^1, E_n^2, \cdots, E_n^r$, and $\hat{\Gamma}_h$ is on a different covering, $H_n^1, H_n^2, \cdots, H_n^s$.*

If M_n is orientable, E_n^i and H_n^j must be oriented concordantly with the fundamental cycle on M_n. Otherwise the work is done modulo 2. It should be noted that Γ_h and $\hat{\Gamma}_h$ are said to be *on the same covering* if the n-cells and their order are the same, and if on each E_n^i the same fundamental complex K_n^i, is used in obtaining Γ_h and $\hat{\Gamma}_h$. Otherwise the coverings are termed different. The proof of Theorem F depends on two lemmas.

LEMMA 1. *If Γ_h is an intersection cycle on the covering $E_n^1, E_n^2, \cdots, E_n^r$ and obtained by small enough deformations, and U_n is another n-cell of some covering of M_n, and $\varepsilon > 0$ is an arbitrary number; then there exists an intersection cycle, Λ_h, on the covering $U_n, E_n^1, E_n^2, \cdots, E_n^r$, and such that $\Gamma_h \sim \Lambda_h$ within ε of G.*

LEMMA 2. *If Γ_h and $\hat{\Gamma}_h$ are both on the covering $E_n^1, E_n^2, \cdots, E_n^r$, and $\varepsilon > 0$ is given; then, if the deformations producing Γ_h and $\hat{\Gamma}_h$ are small enough, $\Gamma_h \sim \hat{\Gamma}_h$ within ε of G.*

28. It will now be shown that Theorem F follows from the lemmas. *If Γ_h is on $E_n^1, E_n^2, \cdots, E_n^r$, and $\hat{\Gamma}_h$ is on H_n^s, $E_n^1, E_n^2, \cdots, E_n^r$; then $\Gamma_h \sim \hat{\Gamma}_h$ with proper stipulations as to size of deformations etc.* By Lemma 1 there is a cycle, Λ_h, on the second covering such that $\Gamma_h \sim \Lambda_h$. Then, again with proper stipulations, Lemma 2 gives $\Lambda_h \sim \hat{\Gamma}_h$; from which follows $\Gamma_h \sim \hat{\Gamma}_h$.

29. *If $\hat{\Gamma}_h$ is on $H_n^1, H_n^2, \cdots, H_n^s$, $E_n^1, E_n^2, \cdots, E_n^r$ and Γ_h is on $E_n^1, E_n^2, \cdots, E_n^r$; then $\Gamma_h \sim \hat{\Gamma}_h$.* This result is obtained by repeated use of the argument of No. 28. But since $H_n^1, H_n^2, \cdots, H_n^s$ covers G, if the deformations are small enough the compound covering is equivalent to $H_n^1, H_n^2, \cdots, H_n^s$ from the point of view of intersections. The statement at the head of this number is thus equivalent to Theorem F.

30. *Proof of Lemma* 1. The chains to be deformed to get Λ_h from Γ_h are A_p^r and A_q^r. These, it should be recalled, are the final deforms of C_p and C_q used in getting Γ_h. Starting with these chains begin, on U_n, to build up Λ_h in the same way that Γ_h was built up from C_p and C_q in f. c. The simplicial piece of Λ_h on U_n will be in part deforms of parts of the A's which gave rise to simplicial pieces of Γ_h. So if the additional deformations are small enough (from Γ_h to Λ_h) and Γ_h itself was got by small enough deformations, L. T., ch. iv shows there are homologies within $\varepsilon/2$ of G between corresponding parts of Γ_h and Λ_h mod neighborhoods, N, of $(h-1)$-complexes on Γ_h of the type $|\, C_p{}^i \cdot C^{*ij}_{q-1} \,|$ (see No. 10). These neighborhoods depend in size on the parts of Γ_h on cells, $E_n{}^k$, $i < k$, and on the additional deformations used to get Λ_h, so they can be arbitrarily small.

If $\Gamma_h{}^u$ is the sum of the closed h-cells of Γ_h on U_n, and N' is a suitable neighborhood of the complex carrying the simplicial part of $F(\Gamma_n{}^u)$; then $\Gamma_h \sim \Lambda_h$ mod $N + N' + (M_n - U_n)$; and the diameter of N' approaches zero as the deformations producing Λ_h from Γ_h approach zero. Outside U_n, Λ_h can be identical with Γ_h so $\Gamma_h \sim \Lambda_h$ mod $(N + N')$. Since $N + N'$ is an arbitrarily small neighborhood of an $(h-1)$-complex, $\Gamma_h \sim \Lambda_h$ as stated in Lemma 1 (see F. M. 2, p. 541).

31. *Proof of Lemma* 2 (see No. 5). The proof involves the construction of an $(h+1)$-chain, C_{h+1} on M_n such that $C_{h+1} \rightarrow \Gamma_h - \hat{\Gamma}_h$ within ε of G. This construction is similar to f. c. and is to be made by induction. In what follows the notation of f. c. will be used for Γ_h. The same notation with a circumflex accent (^) added will be used when $\hat{\Gamma}_h$ is in question.

Step 1. The actual proof proceeds as follows. Choose an $\eta > 0$. Since $A_p{}^1$ and $\hat{A}_p{}^1$ are both deforms of C_p, there is a chain, $W^1{}_{p+1} \rightarrow A_p{}^1 - \hat{A}_p{}^1$ on M_n of mesh $\varepsilon_1/2$. Similarly there is a $(q+1)$-chain, $W^1{}_{q+1} \rightarrow A_q{}^1 - \hat{A}_q{}^1$. By an $\varepsilon_1/2$-deformation carry the closed $(p+1)$-cells of $W^1{}_{p+1}$ entirely on $K_n{}^1$ into a subchain, $C^1{}_{p+1}$, of $K_n{}^1$, leaving $A_p{}^1$ and $\hat{A}_p{}^1$ invariant. This is possible because the requisite parts of the p-chains are already on $K_n{}^1$. Add the deformation chain of the boundary of the piece which was deformed, and call the entire new $(p+1)$-chain $A^1{}_{p+1}$.

Deform $W^1{}_{q+1}$ in the same manner using $K^*{}_n{}^1$ instead of $K_n{}^1$ and leaving invariant all q-cells of $W^1{}_{q+1}$ not in $E_n{}^1$ and not at a distance of more than $4\varepsilon_1$ from $F(E_n{}^1)$, unless they have a point of $C^*{}_q{}^1$ or $\hat{C}^*{}_q{}^1$ on their boundaries in which case they are deformed with the rest. The chains $\hat{C}^*{}_q{}^1$ and $C^*{}_q{}^1$ themselves are to be left invariant. Add the deformation chain of the boundary and call the total deformed chain $A^1{}_{q+1}$ and the part of $A^1{}_{q+1}$ on $K^*{}_n{}^1$, C^{*1}_{q+1}. The chain $X^*{}_q{}^1 = F(C^{*1}_{q+1}) - (C^*{}_q{}^1 - \hat{C}^*{}_q{}^1)$ is then nowhere

nearer than $2\varepsilon_1$ to $F(E_n{}^1)$, whereas $X_p{}^1 = F(C^1{}_{p+1}) - (C_p{}^1 - \hat{C}_p{}^1)$ is nowhere farther than ε_1 from $F(E_n{}^1)$.

Let $$C^1{}_{h+1} = (-1)^{n-q}C^1{}_{p+1} \cdot C^*{}_q{}^1 + \hat{C}_p{}^1 \cdot C^{*1}_{q+1},$$

intersections being of chains on $K_n{}^1$ and $K^*{}_n{}^1$, are here meant in the sense of L. T., ch. iv, § 1. Then if ε_1 is small enough L. T., pp. 169 and 187 give

$$C^1{}_{h+1} \to C_h{}^1 - \hat{C}_h{}^1 + X_h{}^1, \quad \text{where, since} \quad F(\hat{C}_p{}^1) \cdot C^{*1}_{q+1} = 0,$$

$$X_h{}^1 = (-1)^{n-1}C^1{}_{p+1} \cdot F(C^*{}_q{}^1) + \hat{C}_p{}^1 \cdot X^*{}_q{}^1.$$

32. Assume steps $2, 3, \cdots, k-1$ to have been made and the necessary chains to have been defined for i and j, $i < j < k$. Let $X^*{}_q{}^{jk}$ be the chain sum of the closed q-cells of $X^*{}_q{}^j$ which are 1) entirely on $E_n{}^k$, and 2) have no point within $4\varepsilon_1$ of $F(E_n{}^k)$ unless a $(q-1)$-cell of C^{*jk}_{q-1} or $\hat{C}^{*jk}_{q-1}$ is on their boundary, and 3) have no interior points on $X^*{}_q{}^{jg}$ or $Y^*{}_q{}^{ij}$, $g < k$. The chain $X^*{}_q{}^{jk}$ plays the rôle in this proof of C^{*jk}_{q-1} in f. c. It is, roughly speaking, the part of $X^*{}_q{}^j$ in $E_n{}^k$ but not in $E_n{}^j$, and is so defined that all cells of C^{*jk}_{q-1} and $\hat{C}^{*jk}_{q-1}$ are on its boundary.

Step k is then made as follows. As a consequence of step $k-1$ of the induction and step k of f. c., there are on M_n chains W^{k-1}_{p+1} and W^{k-1}_{q+1} such that $W^{k-1}_{s+1} \to A_s{}^k - \hat{A}_s{}^k$ $(s = p, q)$. Subdivide the cells of W^{k-1}_{p+1} and W^{k-1}_{q+1} until their mesh is $\varepsilon_k/2$ and ε_k respectively without altering the A's which already satisfy this condition. Call the new chains sW^{k-1}_{p+1} and sW^{k-1}_{q+1}.

Now carry sW^{k-1}_{p+1} into $A^k{}_{p+1}$ just as $W^1{}_{p+1}$ was carried into $A^1{}_{p+1}$. Call the new $(p+1)$-chain on $K_n{}^k$ $C^k{}_{p+1}$.

Next sW^{k-1}_{q+1} is to be treated.

LEMMA. *If sW^{k-1}_{q+1} contains a $(q+1)$-cell, E_{q+1} having a q-face, E_q, in $X^*{}_q{}^{js}$ or $Y^*{}_q{}^{sj}$, $s < j < k$, and a q-face E'_q in $R_q{}^{k-1}$ (see No. 15), a resubdivision of sW^{k-1}_{q+1} will avoid this without creating new situations of the same type.*

Proof. If E_q and E'_q have a $(q-1)$-face in common, it is, by construction, a $(q-1)$-cell of C^{*js}_{q-1} or D^{*sj}_{q-1}. But by definition of $R_q{}^{k-1}$ this is impossible. So a subdivision of E_{q+1} by section (L. T., p. 68) will avoid the situation in the desired way. A similar lemma holds for $\hat{R}_q{}^{k-1}$.

This lemma shows that if the ε_k-subdivisions are small enough the following definition of R^{k-1}_{q+1} is self-consistent. The chain R^{k-1}_{q+1} is the set of all closed q-cells of sW^{k-1}_{q+1} in $E_n{}^k$ and at a distance of more than $4\varepsilon_1$ from $F(E_n{}^k)$ minus the cells which are

1) in C^{*j}_{q+1}, $j < k$;

2) in Δ^{ij}_{q+1}, the deformation chain joining $X^*{}_q{}^{ij}$ and $Y^*{}_q{}^{ij}$;

plus 3) such q-cells of sW^{k-1}_{q+1} in $E_n{}^k$ but not in 1) or 2) as have on their boundaries,

a) for some j a point of $X^*{}_q{}^{jk}$ but no $(q-1)$-cell of $X^*{}_q{}^{js}$ or $Y^*{}_q{}^{sj}$, $s < k$, (see No. 15),

or b) a q-cell of $R_q{}^{k-1}$ or $\hat{R}_q{}^{k-1}$.

By an ε_k deformation carry R^{k-1}_{q+1} into a sub-chain, C^{*k}_{q+1}, of $K^*{}_n{}^k$. Add the deformation chain of the boundary of the deformed part and call the new chain $A^k{}_{q+1}$. Let $Y^*{}_q{}^{ik}$, $i < k$, be the image under this deformation of $X^*{}_q{}^{ik}$. Take ε_k so small that no cell of $Y^*{}_q{}^{ik}$ is within $2\varepsilon_1$ of $F(E_n{}^k)$. This new chain plays here approximately the rôle of D^{*ik}_{q-1} in f. c. Now the chain $X^*{}_q{}^k = F(C^{*k}_{q+1}) - (C^*{}_q{}^k - \hat{C}^*{}_q{}^k)$ is nowhere nearer than ε_1 to $F(E_n{}^k)$, whereas $X_p{}^k = F(C^k{}_{p+1}) - (C_p{}^k - \hat{C}_p{}^k)$ is nowhere farther than ε_k from $F(E_n{}^k)$.

33. Let $C^k{}_{h+1} = (-1)^{n-q} C^k{}_{p+1} \cdot C^*{}_q{}^k + \hat{C}_p{}^k \cdot C^{*k}_{q+1}$. Then if ε_k is small enough, $C^k{}_{h+1} \rightarrow C_h{}^k - \hat{C}_h{}^k + X_h{}^k$ within $k\eta$ of G, where, since $F(\hat{C}_p{}^k) \cdot C^{*k}_{q+1} = 0$, $X_h{}^k = (-1)^{n-q} C^k{}_{p+1} \cdot F(C^*{}_q{}^k) + \hat{C}_p{}^k \cdot X^*{}_q{}^k$. Define $X_h{}^{ik}$, $Y_h{}^{ik}$, $L_h{}^{ik}$, $i < k$, as follows:

$$X_h{}^{ik} = (-1)^{n-q} C^i{}_{p+1} \cdot C^{*ik}_{q-1} + \hat{C}_p{}^i \cdot X^*{}_q{}^{ik};$$

$$Y_h{}^{ik} = (-1)^{n-q} C^k{}_{p+1} \cdot D^{*ik}_{q-1} + \hat{C}_p{}^k \cdot Y^*{}_q{}^{ik};$$

L^{ik} is the λ_i-neighborhood $(0 < \lambda_i < \eta)$ of

$$|\hat{C}_p{}^i \cdot F(X^*{}_q{}^{ik})| + |\hat{C}_p{}^i \cdot C^{*ik}_{q-1}| + |C_p{}^i \cdot C^{*ik}_{q-1}| + |\hat{C}_p{}^i \cdot \hat{C}^{*ik}_{q-1}|.$$

Take λ_i so large that L^{ik} includes the neighborhoods N^{ik}, M^{ik}, $\hat{N}^{ik}$ and $\hat{M}^{ik}$. The diameter λ_i approaches zero as δ does, (see f. c.). Note that L^{ik} does not depend on $\varepsilon_{i+1}, \varepsilon_{i+2}, \cdots, \varepsilon_k$.

THEOREM G. *If $1 \leq k \leq r$, for every λ_i, the quantities $\varepsilon_{i+1}, \varepsilon_{i+2}, \cdots, \varepsilon_k$ can be taken so small that there exists an $(h+1)$-chain, C^{ik}_{h+1}, within $(k+1)\eta$ of G such that*

$$C^{ik}_{h+1} \rightarrow X_h{}^{ik} - Y_h{}^{ik} \bmod L^{ik}.$$

Proof. The intersections $C^i{}_{p+1} \cdot C^{*ik}_{q-1}$ and $C^k{}_{p+1} \cdot D^{*ik}_{q-1}$ are in the sense of L. T., ch. iv, considering as the original chains $C^i{}_{p+1}$ and C^{*ik}_{q-1} which do not meet one another's boundaries mod L^{ik}, so the proof goes through like Theorem B, No. 18, as far as this pair is concerned. The same sort of proof holds for $\hat{C}_p{}^i$ and $X^*{}_q{}^{ik}$.

34. The construction of the preceeding paragraphs has been so made that all h-cells of X_h^k are either images of h-cells in X_h^i, $i < k$, or within $5\varepsilon_1$ of $F(E_n^k)$. The construction is also such that Theorems C and D (Nos. 22, 23) hold for $C^k_{p+1} \cdot F(C^*{}_q^k)$ and $\hat{C}_p^k \cdot X^*{}_q^k$ and the neighborhoods L^{ij} as they do for $C_p^k \cdot F(C^*{}_q^k)$ and the neighborhoods M^{ij}. This is because the construction of C^k_{p+1} and $X^*{}_q^k$ is exactly analogous to that of C_p^k and $F(C^*{}_q^k)$. Combining these results for $C^k_{p+1} \cdot F(C^*{}_q^k)$ and $\hat{C}_p^k \cdot X^*{}_q^k$ gives:

THEOREM C'. *If* $1 \leq k \leq r$, *all* h-*cells of* X_h^k *which are not cells of* $Y^*{}_q^{ik}$ *are in* L^{ij}, $i < j < k$, *or in some* $X^*{}_q^{ks}$ $k < s \leq r$, *provided the* ε'*s are small enough.*

THEOREM D'. *If an* h-*cell of* X_h^k *is in* $Y^*{}_q^{ik}$ *and* $Y^*{}_q^{jk}$, $i < j < k$, *it is in* L^{ik} *provided the* ε'*s are small enough.*

As before, L^{ij} is always fixed *after* $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_i$ have been determined.

These two theorems combine to give the analogue, E', of Theorem E (No. 24). Theorem E' plus the fact that the process here considered comes to an end at the r-th step, gives $C'_{h+1} \to (C_h^i - \hat{C}_h^i)$ mod ΣL^{ab} within $(r+1)\eta$ of G, where $C'_{h+1} = \Sigma C^i{}_{h+1} - \Sigma C^{ab}_{h+1}$. Since $L^{ab} \supset N^{ab}$ and $\hat{N}^{ab}$, $C'_{h+1} \to \Gamma_h - \hat{\Gamma}_h + Q_h$, where Q_h is an h-cycle on ΣL^{ab}. If $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r$ are small enough, $Q_h \sim 0$ within $(r+2)\eta$ of G, so there is an $(h+1)$-chain

$$C_{h+1} \to \Gamma_h - \hat{\Gamma}_h$$

within $(r+2)\eta$ of G. If $\eta = \varepsilon/(r+2)$ this proves Lemma 2, and completes the program outlined in Nos. 1-5.

CORNELL UNIVERSITY.

ON THE IMBEDDING OF METRIC SETS IN EUCLIDEAN SPACE.*

By W. A. Wilson.

1. It is the purpose of this note to make a slight extension of results previously obtained by the writer † and to give a modification of Menger's general conditions for the imbedding of n points of a metric space in Euclidean space.

With regard to the first topic it is proved on pp. 515-16 of the paper mentioned that a complete space, which is convex and externally convex and has the four-point property, has the n-point property for every integer n. We now proceed to show that the requirement of external convexity is needless.

Using the notation of this proof, let $T_1 \simeq T'_1$, $T_0 \simeq T'_0$, and $T_{01} \simeq T'_{01}$. If the line through a'_0 and a'_1 meets T'_{01}, external convexity is not needed for the proof as given.‡ In the opposite case it is clear that there is a point u' in T'_1 near enough to the centroid of T'_{01} so that: if a'_0 and a'_1 are on the same side of E_{n-2} and a'_0u' is produced to meet T'_{01} in x', $u'x'$ lies also in T'_0; and, if a'_0 and a'_1 are on opposite sides of E_{n-2}, a'_1u' cuts T'_{01}. In the congruence $T_1 \simeq T'_1$, let $u \sim u'$. Then by the argument of p. 516 the points $u, a_1, a_2, \cdots, a_n$ can be imbedded in E_n and the sign of the determinant $D(u, a_1, a_2, \cdots, a_n) \neq \operatorname{sign}(-1)^n$.

Let us now suppose that $\operatorname{sign} D(a_0, a_1, \cdots, a_n) = \operatorname{sign}(-1)^n$. Since D is a continuous function of each variable and changes sign when a_0 is replaced by u, there is some point v on the segment ua_0 for which $D(v, a_1, a_2, \cdots, a_n) = 0$. Then the points $v, a_1, a_2, \cdots, a_n$ can be imbedded in E_{n-1} so that $T_0 \simeq T'_0$ in one of two ways: (1) v' on the same side of E_{n-1} as a'_1; (2) v' on the opposite side.

In the first case, since v lies within T_1, the congruence $v + a_2 + \cdots + a_n \simeq v' + a'_2 + \cdots + a'_n$, which is a sub-congruence of $v + a_1 + \cdots + a_n \simeq v' + a'_1 + \cdots + a'_n$, defined in the preceding paragraph, is also a sub-congruence of $T_1 \simeq T'_1$, which includes $T_{01} \simeq T'_{01}$.‡ If $x \sim x'$ in the congruence $T_1 \simeq T'_1$, we have $a_0 + v + x \simeq a'_0 + v' + x'$ and (by the previous paragraph) $v + x + a_1 \simeq v' + x' + a'_1$. Now by the four-point property

* Presented by title to the Society, September, 1934.

† "A relation between metric and Euclidean spaces," *American Journal of Mathematics*, vol. 54 (1932), pp. 505-517.

‡ For we can then refer to Theorem I of § 8 instead of Theorem IV.

$a_0 + v + x + a_1 \simeq a''_0 + v' + x' + a'_1$, where a''_0 is some point of E_{n-1}. These congruences combined give $v'a'_0 = va_0 = v'a''_0$ and $x'a'_0 = xa_0 = x'a''_0$, while $x'a'_0 = x'v' + v'a'_0$ and $x'a''_0 = x'v' + v'a''_0$. Hence $a'_0 = a''_0$ and so $a_0a_1 = a'_0a'_1$.

Precisely the same argument applies when v' and a'_1 are on opposite sides of E_{n-2}. Thus the assumption that sign $D(a_0, a_1, \cdots, a_n) =$ sign $(-1)^n$ is false, since it has led to the contradiction that $a_0, a_1, \cdots, a_n$ can be imbedded in E_{n-1}. That is, the theorem in question is valid when external convexity is not given.

It follows, therefore, that the theorem of § 12 (*loc. cit.*) can be modified to read: *A convex complete separable space which has the four-point property is congruent with a sub-set of some E_n or of Hilbert space.**

2. Turning to the second topic, we recall Menger's condition † for imbedding $n + 1$ points of a metric space in E_n, namely that, if the distances between the respective pairs of any $k + 1$ $(k \leqq n)$ of these points are substituted in the formula for the volume of a k-dimensional simplex in terms of the edges, the result is real. This condition can be put into another form which is of some interest.

Let the $n + 1$ points be designated by the integers $0, 1, 2, \cdots, n$; then 01, 02, etc., will denote segments or lengths of segments. Assuming for the moment that the points can be imbedded in E_n, let $0 : rs$ denote the angle between the segments $0r$ and $0s$. Then

$$(rs)^2 = (0r)^2 + (0s)^2 - 2(0r)(0s) \cos 0 : rs. \tag{1}$$

For four points 0, 1, r, s, which are the vertices of a tetrahedron let $01 : rs$ denote the dihedral angle of edge 01 and faces $01r$ and $01s$. It is well known that

$$\cos 0 : rs = \cos 0 : 1r \cos 0 : 1s + \sin 0 : 1r \sin 0 : 1s \cos 01 : rs. \tag{2}$$

In general, if $0, 1, \cdots, k+1, r, s$, are the vertices of a $k + 3$ dimensional simplex, let $01 \cdots k+1 : rs$ denote the space-angle having the "edge" $01 \cdots k+1$ and the "faces" $01 \cdots k+1, r$ and $01 \cdots k+1, s$. (This is the angle between two $k + 2$-dimensional spaces in a $k + 3$-dimensional

* The referee states that the reasoning employed above can be applied with little change to the corresponding work of L. Blumenthal, "Concerning spherical spaces," *American Journal of Mathematics*, vol. 57 (1935), pp. 51-61. See Theorems 3.3, 4.1, and 4.2. The property of external convexity corresponds to that of being "diameterized" in spherical spaces.

† "Untersuchungen über allgemeine Metrik, II," *Mathematische Annalen*, vol. 100, pp. 133 and 136.

space.) The spherical cosine law is also valid for these generalizations of dihedral angles, giving *

$$\begin{aligned}(3)\qquad \cos 01\cdots k:rs &= \cos 01\cdots k:k+1,r \cos 01\cdots k:k+1,s\\ &\quad+\sin 01\cdots k:k+1,r \sin 01\cdots k:k+1,s \cos 01\cdots k+1:rs.\end{aligned}$$

Now, if V is the volume of the simplex $(012\cdots n)$, it is known † that the formula used by Menger can be transformed with the aid of (1) into

$$V^2=\frac{(01)^2(01)^2\cdots(0n)^2}{(n!)^2}\cdot\Delta_n,$$

$$\text{where}\qquad \Delta_n=\begin{vmatrix} 1 & \cos 0:12 & \cos 0:13 & \cdots & \cos 0:1n\\ \cos 0:12 & 1 & \cos 0:23 & \cdots & \cos 0:2n\\ \cos 0:13 & \cos 0:23 & 1 & \cdots & \cos 0:3n\\ \cdot & \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdot & & \cdot\\ \cos 0:1n & \cos 0:2n & \cos 0:3n & \cdots & 1 \end{vmatrix}.$$

Multiply the first column successively by $\cos 0:12$, $\cos 0:13$, etc., and subtract from the columns headed by these factors. Clearly $1-\cos^2 0:1s=\sin^2 0:1s$. We also get terms of the form $\cos 0:rs-\cos 0:1r\cos 0:1s$. In such cases substitute $\sin 0:1r \sin 0:1s \cos 01:rs$ by means of the spherical cosine law (2). We can then remove common factors and get

$$\Delta_n=\sin^2(0:12)\sin^2(0:13)\cdots\sin^2(0:1n)\cdot\Delta_{n-1},$$

$$\text{where}\ \Delta_{n-1}=\begin{vmatrix} 1 & \cos 01:23 & \cos 01:24 & \cdots & \cos 01:2n\\ \cos 01:23 & 1 & \cos 01:34 & \cdots & \cos 01:3n\\ \cos 01:24 & \cos 01:34 & 1 & \cdots & \cos 01:4n\\ \cdot & \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdot & & \cdot\\ \cos 01:2n & \cos 01:3n & \cos 01:4n & \cdots & 1 \end{vmatrix}.$$

We treat this determinant as we did Δ_n, using formula (3) above for the case that $k=1$ and we get

$$\Delta_{n-1}=\sin^2(01:23)\sin^2(01:24)\cdots\sin^2(01:2n)\cdot\Delta_{n-2},$$

* This is given by James McMahon, "Hyperspherical goniometry and its application to correlation theory for n variables," *Biometrika*, vol. 15 (1923), p. 187. It can also be deduced from a result of Ernst Liers, "Über den Inhalt des vier dimensionalen Pentaeders," *Archiv der Mathematik und Physik*, 2d Series, vol. 12, pp. 344-351.

† See Study, *Zeitschrift für Mathematik und Physik*, vol. 27, p. 150.

where Δ_{n-2} has $n-2$ rows and columns and its elements are cosines of the space-angles $012:rs$.

Continuing in the same fashion, we finally reach the relation

$$\Delta_4 = \sin^2(012\cdots n-4:n-3,n-2)\sin^2(012\cdots n-4:n-3,n-1)$$
$$\sin^2(012\cdots n-4:n-3,n)\cdot\Delta_3,$$

where

$$\Delta_3 = \begin{vmatrix} 1 & \cos 012\cdots n-3:n-2,n-1 & \cos 012\cdots n-3:n-2,n \\ \cos 012\cdots n-3:n-2,n-1 & 1 & \cos 012\cdots n-3:n-1,n \\ \cos 012\cdots n-3:n-2,n & \cos 012\cdots n-3:n-1,n & 1 \end{vmatrix}$$
$$= \sin^2(012\cdots n-3:n-2,n-1)\sin^2(012\cdots n-3:n-2,n)\sin^2(012\cdots n-2:n-1,n).$$

This reduction expresses V^2 as the product of non-negative factors. It follows, then, for $n+1$ points in any metric space, that V is real if formulas (1), (2), and (3) above define real angles and the reduction can be carried to the end. Looking back, we observe that the angles $0:rs$ defined by the plane cosine law always have definite real values, since the space is metric. The successive space-angles $012\cdots k+1:rs$ $(k=0,1,\cdots,n-3)$, are defined by the spherical cosine law (3), which can be written

$$\cos 012\cdots k+1:rs$$
$$= \frac{\cos 012\cdots k:rs - \cos 012\cdots k:k+1,r\cos 012\cdots k:k+1,s}{\sin 012\cdots k:k+1,r\sin 012\cdots k:k+1,s}.$$

The space-angle thus defined has a definite real value unless the absolute value of the fraction is greater than 1 or the denominator is zero.

Let us now assume the following postulates for angles of all orders:

I. $012\cdots k:rs + 012\cdots k:rt + 012\cdots k:st \leqq 2\pi$;

II. $|\,012\cdots k:rs - 012\cdots k:rt\,| \leqq 012\cdots k:st \leqq 012\cdots k:rs + 012\cdots k:rt.$

If $\sin 012\cdots k:k+1,r=0$, this angle is 0 or π. It then follows from these postulates that angles $012\cdots k:k+1,t$ and $012\cdots k:rt$ are equal or supplementary for every value of t. In that event the columns of Δ_{n-k} which contain $\cos 012\cdots k:k+1,r$ are identical or one is the negative of the other. In both cases $\Delta_{n-k}=0$ and the introduction of higher space-angles is unnecessary, as $V^2=0$.

If neither $\sin 012\cdots k:k+1,r$ nor $\sin 012\cdots k:k+1,s$ is zero, we can easily show by elementary trigonometry that $1+\cos 012\cdots k+1:rs \geqq 0$ and $1-\cos 012\cdots k+1:rs \geqq 0$; whence $012\cdots k+1:rs$ has a definite real value between 0 and π. Thus, if the above angle postulates hold for

every k, either $V^2 = 0$ or the reduction of the determinants can be carried out to the end.

We can then state as a theorem that $n + 1$ *given points of a metric space can be imbedded in* E_n *unless there is some set of* $k + 3$ *points,* $1 \leqq k \leqq n - 2$, *determining three angles or space-angles* $a_1a_2 \cdots a_k : a_ra_s$, $a_1a_2 \cdots a_k : a_ra_t$, $a\,a \cdots a_k : a_sa_t$ *such that their sum is greater than* 2π *or the metric triangle inequality fails.*

The reader will note that this result is an extension of Blumenthal's theorem on the equivalence of the four-point property and Postulates I and II for plane angles.* Neither result is any simpler to apply to a metric space defined by the distances between the respective pairs of points than are Menger's criteria, but both have some interest as showing that the determinant criteria may be regarded as phases of the triangle inequality.

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* L. M. Blumenthal, "A note on the four point property," *Bulletin of the American Mathematical Society*, vol. 39, pp. 423-426. It may be remarked in the case of four points that the condition that $0 : ab + 0 : ac + 0 : bc \leqq 2\pi$ is unnecessary, as a failure of this at any vertex involves a failure of the metric triangle inequality at some other vertex.

ON SEMICOMPACT SPACES.†

By Leo Zippin.

1. *Introduction.* It is a well established idea in topology to consider spaces in which certain general properties are assumed to hold only locally, and it is not new to go a step further and transfer these properties from neighborhoods to boundaries of neighborhoods. None the less the fundamental notion of compactness does not appear to have been treated from this point of view. It is the object of this paper to prove two theorems, based on a property we call semicompactness, which seem to us not uninteresting.

Definition. A topologic (Hausdorff) space C is called *semicompact at a point* x if every neighborhood U_x contains a V_x such that $B(V_x)$, the boundary of V_x, is compact. It is called *semicompact* if it has this property at every point. Of course $B(V_x)$ is necessarily closed so that it is actually self-compact.

1.1. Now we have allowed that $B(V_x)$ may be vacuous. Therefore it is clear, for example, that every zero-dimensional topologic space is semicompact. This suggests, at least, that this concept is hardly likely to be very fruitful without some restriction on the nature of the topologic space C. In this paper we shall go quite a way in delimiting the class of spaces we consider. We shall require of C that it be a separable, complete metric space; that is to say that C can be metrized in such a way that every Cauchy sequence converges. In these spaces it will transpire that the notion of semicompactness is strikingly near to that of local compactness. A semicompact C which is separable and complete metric will be called, for shortness, an s. C.-space.

1.2. The two theorems of this paper are concerned with the possible compactifications of an s. C.-space. Thus while a locally compact separable metric space can be compactified by the addition of a single "point," an s. C.-space may always be compactified by the addition of a countable set (Theorem I). In Theorem II we demand that the s. C.-space be connected and locally connected and obtain a considerable generalization of a Theorem of Freudenthal.‡ We shall conclude with a few remarks on, and some applications of, this theorem.

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† Presented to the American Mathematical Society December, 1933. See Abstract, *Bulletin of the American Mathematical Society*, vol. 40 (1934), p. 56, no. 97.

‡ See § 6 and note thereto. We were not aware of Freudenthal's paper at the time of publication of the Abstract for this paper.

2. THEOREM I. *Every s. C.-space C may be compactified by the addition of a countable point-set.*

We may suppose that C is not compact, otherwise the theorem is trivial. We shall associate with C a metric in which every Cauchy sequence converges.

Definition. An open subset V of C such that $B(V)$, the boundary of V, is compact will be called a *domain*: an *ϵ-domain* if its diameter $< \epsilon$.†

2. 1. *The ϵ-Partition.* Let V denote any non-compact domain of C, e. g. C itself. Then $\bar{V}$ is complete in our chosen metric and there must exist an $\epsilon' > 0$ such that $\bar{V}$ is not the sum of any finite number of subsets of diameter $< \epsilon'$.‡ Any positive number $< \epsilon'$ will be called suitable for our Partition. Choose some fixed ϵ, $0 < \epsilon < \epsilon'$. From the separability of C and its semi-compactness, there exists a sequence of *ϵ-domains*, $U_1, U_2, \cdots$, which cover $\bar{V}$. Let $K_1 = V \cdot U_1$ and, generally, $K_n = V \cdot (U_n - \sum_1^{n-1} \bar{U}_i)$.

2. 2. We assert that the K_n are *ϵ-domains.*§ It is obvious that they are small enough, and open. We must show that $B(K_n)$ is compact. It is if it is vacuous. If it is not vacuous let $x \subset B(K_n) \subset \bar{K}_n \subset \bar{V} \cdot \bar{U}_n$.¶ Then $x \not\subset \sum_1^{n-1} U_i$ which is open and contains no point of K_n. Then if $x \subset \sum_1^{n-1} \bar{U}_i$, $x \subset \bar{U}_k - U_k = B(U_k)$ for some $k < n$. On the other hand if $x \not\subset \sum_1^{n-1} \bar{U}_i$, then either $x \not\subset U_n$ or $x \not\subset V$. For if $x \subset V \cdot U_n$, it follows that $x \subset K_n$: by assumption, however, $x \subset B(K_n)$. Therefore since $x \subset \bar{V} \cdot \bar{U}_n$, it follows that $x \subset \bar{V} - V = B(V)$, or else $x \subset \bar{U}_n - U_n = B(U_n)$. Then $B(K_n) \subset B(V) + \sum_1^n B(U_i)$, and this sum is compact.

2. 3. It is important for us to notice that although the $\bar{K}_n$ are not open they "cover" $\bar{V}$ in a very definite sense. Let z be any point of $\bar{V}$ and n the least integer such that $z \subset U_n$. Let $z_1, z_2, \cdots$, be any sequence of points of $\bar{V}$ converging to z. Without loss of generality we may suppose them in U_n. Let y denote an arbitrary one of the points $z, z_1, z_2, \cdots$, and let $y_1, y_2, \cdots$, denote a sequence of points of V converging to y. We may assume that these points

† This is a slight departure from customary terminology, which we emphasize by italics.

‡ Otherwise $\bar{V}$ would be compact. See Hausdorff, *Mengenlehre*, 2nd Ed., p. 108.

§ We agree that the null-set is open, therefore an *ϵ-domain.*

¶ Here, as in the sequel, x denotes any point of the space, restricted in so far only as is immediately made evident.

also are in U_n. Now let x denote an arbitrary one of the points $y_1, y_2, \cdots$, and let m be the *least* integer such that $x \subset \bar{U}_m$. This integer depends on x, of course, but for every choice of x, $m \leqq n$. Finally, let $x_1, x_2, \cdots$, denote a sequence of points of U_m converging to x. All but a finite number of these are in V, and at most a finite number of them can belong to $\sum_{i=1}^{m-1} \bar{U}_i$. Therefore almost all of them belong to K_m and consequently $x \subset \bar{K}_m$. Then it is clear that for every integer k, $y_k \subset \sum_{i=1}^{m} \bar{K}_i$. But then y too belongs to this set, and this means that every one of the points $z, z_1, z_2, \cdots$, belongs to it. What we have proved can be expressed as follows: to every point z of $\bar{V}$ there exists an n such that z is an inner point (relative to $\bar{V}$) of $\sum_{i=1}^{n} \bar{K}_i$.

2.4. Let $O_n = \bar{V} - \sum_1^n \bar{K}_i$. It is clear that the O_n form a monotonic sequence, i. e. $O_n \supset O_{n+1}$, whose product is vacuous, and that each is open relative to $\bar{V}$. It is easily seen † that $B(O_n)$ is compact. Now since $B(V)$ is compact and closed, there is an integer n such that $B(V)$ is a subset of inner points, relative to $\bar{V}$, of $\sum_1^n \bar{K}_i$, i. e. no point of $B(V)$ is a limit point of $\bar{V} - \sum_1^n \bar{K}_i = O_n$. This follows from the concluding remark of the previous section by an application of the Heine-Borel Theorem. Now $B(V) \cdot \bar{O}_n = 0$ implies $\bar{O}_n \subset V$. Let D_1 denote the first O_m such that $\bar{D}_1 \subset V$. Then D_1 is a *domain* of C. Let D_2 denote the first O_m thereafter, such that $\bar{D}_2 \subset D_1$. It is clear that we can find a subsequence D_m of the O_n such that:

$$\bar{V} \supset V \supset \bar{D}_1 \supset D_1 \supset \bar{D}_2 \supset D_2 \supset \cdots.$$

It is clear that each D_n is a *domain* of C and that $\Pi \bar{D}_n = 0$. Notice that no D_n is vacuous, since $\bar{V} \not\subset \sum_1^N \bar{K}_i$ for any integer N, by our choice of ϵ. The sequence of *cells* K_n will be called an ϵ-partition of $\bar{V}$. The corresponding sequence D_m will be said to define an *ideal point* associated with this partition.

3. *The ideal points of* C. For notation's sake, we write $C = C_1{}^0$. Let us make an ϵ_0-partition of $C_1{}^0$ for a suitable ‡ $\epsilon_0 < 1$. We designate by $P_1{}^0$ the associated ideal point and by $D^0{}_{1,m}$, $(m = 1, 2, \cdots)$, the *domains* defining $P_1{}^0$. Now let $C_1{}^1, C_2{}^1, \cdots$, denote those of the cells of this partition which

† Compare § 2. 2.
‡ See § 2. 1.

are not compact. If there are any, we make an ϵ_{1n}-partition of each $C_n{}^1$ for a suitable ϵ_{1n} (which varies with the cell) $< \frac{1}{2}$. This is possible since each cell is a *domain.* The associated ideal point is denoted by $P_n{}^1$, its defining *domains* by $D^1{}_{n,m}$, $(m = 1, 2, \cdots)$. Now let $C_1{}^2, C_2{}^2, \cdots$, denote those cells which are not compact which result from any one of the countable set of preceding partitions: their totality is at most countable. Each of these, if any exist, is subjected to an $\epsilon_{2,n}$-partition, every $\epsilon_{2,n} < 1/4$.

Now we may arrive at an integer N such that all the cells confronting us after the N-th partition are compact.† In this event the process will be terminated and no further ideal points introduced. Otherwise we continue the partitioning indefinitely, every non-compact cell $C_m{}^N$ of the N-th stage being $\epsilon_{N,m}$-partitioned for a suitable $\epsilon_{N,m}$ (depending on the cell) $< 1/2^N$, $(N = 1, 2, 3, \cdots)$.

Whichever of the above alternatives we face, it is clear that we have introduced an at most countable set P of ideal points where each one is some $P_m{}^N$ in our construction,‡ being associated with a cell $C_m{}^N$, diam. $(C_m{}^N) < 1/2^N$. The point $P_m{}^N$ is defined by a properly monotonic sequence, $D^N{}_{m,n}$, $(n = 1, 2, \cdots)$, of domains of $C_m{}^N$, $\prod_{n=1}^{\infty} \bar{D}^N{}_{m,n} = 0$.

3.1. Now let C'' denote the abstract "point-set" $C + P$ topologized as follows. Let $G_1, G_2, \cdots$, denote a sequence of *domains* of C which generates § the space C and which includes every defining *domain* for every ideal point $P_m{}^N$ in P. Now if $P_m{}^N$ is any ideal point of C, and G_k any domain of the sequence, we shall say that $P_m{}^N$ *belongs* to G_k if this set contains any one of the defining *domains* for $P_m{}^N$ (in which case, of course, it contains almost all of them). Now let $G_1'', G_2'', \cdots$, denote the point sets G_n to which have been added all the ideal points belonging to them. By definition, each G_n'' is a neighborhood of every point of C'' which it contains. Let us observe at once that $G_m \cdot G_n = 0$ (in C) implies $G_m'' \cdot G_n'' = 0$ (in C''). This is obvious, for if $G_m'' \cdot G_n''$ contained an ideal point it would have to contain a non-vacuous *domain* of C, and if it contained a point of C this would have to be a point of $G_m \cdot G_n$.

3.2. It is trivial that every point of C'' belongs to at least one neighborhood of the system and that if it belongs to two neighborhoods it must belong

† Actually this cannot happen unless $C_1{}^0 = C$ is locally compact, but that is immaterial to the proof.

‡ The ranges of N and of m in its dependence on N depend on the particular choice of partitions.

§ i. e. is a *basis* for the neighborhoods of C.

to a neighborhood common to both of them. Then, since each G_n'' is a neighborhood of every one of its points, we merely have to show that if x and y are distinct points of C'', there exist $G_m'' \supset x$, $G_n'' \supset y$, $G_m'' \cdot G_n'' = 0$, in order to conclude that C'' is a Hausdorff space. This is trivial, excepting possibly in the case that x is an ideal point P_k^N and y is some $P_{k'}^{N'}$ of P. Here we may suppose, on symmetry, that $N \leqq N'$. Now P_k^N is associated with the partition of a non-compact cell C_k^N and $P_{k'}^{N'}$ with that of $C_{k'}^{N'}$. If $N' = N$, $k' = k$, the two points are not distinct. If $N' = N$, $k' \neq k$, P_k^N belongs to a domain $G_m \subset C_k^N$ and $P_{k'}^N$ belongs to a domain $G_n \subset C_{k'}^N$ and $G_m \cdot G_n \subset C_k^N \cdot C_{k'}^N = 0$. On the other hand, if $N' > N$, $C_{k'}^{N'} \subset C_h^N$ for some h. If $h \neq k$ we have the same situation as above. If $h = k$ then $C_{k'}^{N'} = C_j^{N+1} \subset C_k^N$. But then there is a domain G_m to which P_k^N belongs such that $G_m \cdot C_j^{N+1} = 0$ and $P_k^{N'}$ belongs to a subdomain G_n of C_j^{N+1}. Therefore, in view of the last remark of § 3. 1, we have been able, whichever of the cases above may have arisen, to find $G_m'' \supset P_k^N = x$ and $G_n'' \supset P_{k'}^{N'} = y$ such that $G_m'' \cdot G_n'' = 0$. Therefore C'' is certainly a Hausdorff space. It is trivial that C'' is completely separable (i. e. has a countable neighborhood basis). It is clear that C may now be regarded as a topologic subspace of C'', if we ignore the convenient metric we have attached to it.

3. 3. Let us prove finally, that C'' is compact.† To this end, let $x_1, x_2, \cdots$ denote any sequence of points of C''.

i) If there exists any integer N such that infinitely many of the cells resulting from the first N partitions contain at least one point of the sequence, then there is a first such N. Then the sequence (x_n) has an infinite subsequence in some C_m^{N-1} and has at least one point in common with every $D_{m,n}^{N-1}$, $(n = 1, 2, \cdots)$. Therefore, in this case, the ideal point P_m^{N-1} is a limit point (not necessarily the only one) of the sequence (x_n).

ii) If there is any N such that an infinite subsequence of (x_n) belong to a compact cell (in C) or belong to the boundary (which is compact) of any cell of the N-th partitions, then the subsequence consists of points of C which have at least one limit point in C and this is also a limit point of the given subsequence, in C''.

iii) Finally, if neither of the previous cases ever arises, it is easy to see that we can find a monotonic sequence of non-compact cells,

$$C = C_1^0, C_{n_1}^1, C_{n_2}^2, \cdots,$$

such that for every $C^m_{n_m}$ there is at least one point x_{k_m} of our sequence which

† We may suppose all topologic notions defined for C'', as customarily.

belongs to it. But by our construction, the diameters of these cells converge to zero. Then it is a well known consequence of the completeness of C (our metric exhibiting this completeness) that there is a unique point of C common to the closures of these cells, and it is clear that this point is a limit point in C'' of the given sequence.

Therefore C'' is a compact, completely separable Hausdorff space and, as is well known, metrizable. We have observed that our space C is topologically equivalent to a subset C of $C'' = C + P$ where P is countable and $P \subset \bar{C} = C''$. Then C'' is a compactification of C and Theorem I is proved.

4. We may remark that Theorem I is characteristic of s. C.-spaces. This follows from the simple

THEOREM. *If C'' is any compact, metrizable space and $C = C'' - Q$ where Q is any totally disconnected F_σ,*† *then C is an s. C.-space.*

It is clear that C must be separable metric, and well known (Alexandroff) that it is complete in some metric. We merely have to show that it is semicompact. This will follow if we can show that, under our hypotheses, every point of C'' has arbitrarily small neighborhoods whose boundaries are vacuous relative to Q. Write $Q = \Sigma Q_n$, where Q_n is closed, and totally disconnected. Therefore Q_n is zero dimensional in the Menger-Urysohn sense.

4. 1. Then if x denotes any point of C'', $x + Q$ is a zero dimensional point-set.‡ Therefore, for any fixed $\epsilon > 0$ we can write $x + Q = H_1 + H_2$, $H_1 \cdot H_2 = 0$, where $H_1 \supset x$, diam.$(H_1) < \epsilon/3$, and both sets are closed in $x + Q$. Now cover every point y of H_1 by an open set D_y (of C'') $0 <$ diam.$(D_y) <$ Min.$[1/3\epsilon, 1/2$ dist.$(y, \bar{H}_2)]$, and let $D = \sum_y D_y$. It is clear that $0 <$ diam.$(D) < \epsilon$, and that $H_1 \subset D$ which is open. It is easy to see that $H_2 \cdot \bar{D} = 0$.§ Then the boundary of D cannot contain any point of Q so that D is the desired neighborhood, and the theorem is proved.

We need hardly remark that it is not necessary that an F_σ subset Q of a compact metrizable C'' be totally disconnected in order that $C = C'' - Q$ shall be an s. C.-space.

† $Q = \Sigma Q_n$, Q_n closed. This includes the case that Q is countable.

‡ We are appealing to the "Summensatz" of dimension-theory. A proof of what we need can be carried through by a method which Menger has called "Methode der Modification der Umgebungen in der Nähe ihr Begrenzungen" and on which his proof of the Summensatz rests. See his book *Dimensionstheorie*, p. 94.

§ Compare the lemma of Urysohn, "Sur les multiplicités Cantoriennes," *Fundamenta Mathematicae*, vol. 7 (1925), p. 69.

5. It is clear that if an s. C.-space C is connected, the C'' of Theorem I is also connected. If, further, C is locally connected, we may suppose that those G_n (of § 3. 1) which generate C were chosen as connected point-sets and the corresponding G_n'' will be connected. Then, in this case, C'' will certainly be locally connected at every point of C. Consequently, by a theorem of Mazurkiewicz, C'' will be locally connected since $C'' - C = P$ is totally disconnected.

Definition. A connected and locally connected s. C.-space will be called *semipeanian.*†

5. 1. We have just proved the

COROLLARY. *A semipeanian C is topologically contained in a peanian $C'' = \bar{C} = C + P$, P countable.*

Now there are many possible compactifications of C. If we require that the set of ideal points which we adjoin shall be at most countable, then there is not any C'' which is invariantly associated with a general C.‡ Moreover, in this case, the ideal points of C will, in general, "interrupt" C'', in the sense, for example, that it may not be possible to join two neighboring points of C by a small arc of C'' which avoids P. If we do not insist on compactifying C with a countable point-set, then we can show that there exists a peanian C^* invariantly associated with C and rather simply related to it. We may say that the ideal points offer a minimum of interruption. The sense of this will be made precise in Theorem II.

Definition. A totally disconnected subset Q of a peanian C^* will be called totally avoidable provided that $D - D \cdot Q$ is connected for every open connected subset D of C^*.§

6. THEOREM II.¶ *Every semipeanian C is topologically contained in a*

† Complete-metric, separable, connected and locally connected spaces are commonly called quasipeanian. Thus, semipeanian = quasipeanian + semicompact. Compact, metr., con. and loc. con. spaces we shall call peanian.

‡ We shall return to this in § 7.

§ This is a special case of a more general definition of total avoidability, due to Wilder.

¶ We have already remarked that this is a Theorem of Freudenthal in the case that C is locally compact. See H. Freudenthal, "Uber die Enden topologischer Raume und Gruppen," *Mathematische Zeitshcrift*, vol. 33 (1931). Satz 7, p. 702. A similar compactification was used by us in characterizing subsets of a simple closed surface which we called cylinder-trees. See "Study of continuous curves . . . ," *Transactions of the American Mathematical Society*, vol. 31 (1929), Theorem 6, p. 763. However

uniquely determined peanian $C^* = \bar{C}$ such that $Q = C^* - C$ is a totally disconnected and totally avoidable F_σ.

Proof. By the corollary of § 5. 7, C may be compactified to a peanian $C'' = C + P$, P countable.† Let U_n'', $(n = 1, 2, \cdots)$, be a null-sequence ‡ of open connected subsets generating C'' such that $P \cdot B(U_n'') = 0$.§ If U'' denotes any U_n'', $U = C \cdot U''$, and x is any point of U, then $U \supset U_x \supset x$ where U_x is connected and open in C. This is an immediate consequence of the fact that C is topologically contained in C'', and is locally connected. It follows at once that the set of components of U is at most countable.

6. 1. Although we do not need it at this moment it is convenient to prove now that if p is any point of $P \cdot U''$ and $2\epsilon = \text{dist.}\{p, B(U'')\}$ there are only a finite number of components of $U = C \cdot U''$ which meet $S(p, \epsilon)$.¶ For if x is any point of $U \cdot S(p, \epsilon)$ any y denotes any point of $C - U$, the existence of an arc xy of C shows that the component $U_x \supset x$, of U, has at least one point on the boundary of every $S(p, \epsilon')$ where $\epsilon < \epsilon' < 2\epsilon$. Then if there were infinitely many components in question, there would exist at least one point $x_{\epsilon'}$ on $B\{S(p, \epsilon')\}$ which was a limit point of points of distinct components. Now $x_{\epsilon'} \not\subset U$, since the components are open in U. Therefore $x_{\epsilon'} \subset P$. But this is impossible since the $x_{\epsilon'}$ are distinct for different ϵ', and P is at most countable.

6. 2. *Ideal points of C.* The totality of components of $C \cdot U_n''$, $(n = 1, 2, \cdots)$, is countable, by the last remark of § 6. We denote them, in some simple order by $W_1, W_2, \cdots$. A monotonic sequence W_{n_i}, $(i = 1, 2, \cdots)$, of sets W_m will be called a *proper sequence* if the product of their closures (in C'') is a single point of P. Two proper sequences W_{n_i}, W_{m_i}, $(i = 1, 2, \cdots)$, are called *equivalent* if for every j there is a k such that $W_{n_j} \supset W_{m_k}$ and conversely to every k a j such that $W_{m_k} \supset W_{n_j}$. It is trivial that our definition satisfies the usual conditions for equivalence. A class of equivalent proper sequences will be called an *ideal point* of C. It is clear that with each ideal point of C there is associated a unique point of P, this correspondence being, in general, many-one. The totality of ideal points we denote by Q. We shall

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this process is there carried out in a very special case and its essential generality was not then suspected by us. Our method there, as here, differs from Herr Freudenthal's in that we exploit a preliminary compactification of the space.

† The use of C'' is a pure convenience to facilitate the handling of the ideal points which we presently define.

‡ i. e. diam. (U_n'') converges to zero in an arbitrary fixed metric for C''.

§ This condition is easily fulfilled. See § 4. 1.

¶ The set of points whose distance from p is $< \epsilon$.

say that an ideal point q *belongs* to a set W of C if $W \supset W_{n_k}$, where W_{n_k} is any set in any proper sequence defining q. It is clear that W contains almost all the sets in any equivalent proper sequence. Observe that if q is an ideal point, p the associated point of C'', W'' any neighborhood of p in C'' and $W = C \cdot W''$, then q belongs to W.

6.3. *The space C^*.* Let C^* denote the abstract point set consisting of points and ideal points of C. We may write this $C^* = C + Q$. Let W^*_n denote the subset of C^* consisting of all points of $W_n \subset C$ and all points of Q which *belong* to W_n by the definition of the preceding section. We shall topologize C^* by agreeing that W^*_n, $(n = 1, 2, \cdots)$, is a *neighborhood* of every one of its points. We observe that $W_m \cdot W_n = 0$ (in C) implies $W^*_m \cdot W^*_n = 0$ (in C^*).† It is trivial that these neighborhoods have all the Hausdorff properties with the possible exception of this one: that if x^* and y^* are distinct points of C^* there exist $W^*_m \supset x^*$, $W^*_n \supset y^*$, $W^*_m \cdot W^*_n = 0$. This is also trivial in the case that the points x and y of C'' associated ‡ with x^* and y^* are distinct, in view of the observations above. We shall dispose of the remaining case in § 6.5.

6.4. Let us suppose that q is an ideal point of C and that every W_{n_j}, $(j = 1, 2, \cdots)$, of any corresponding proper sequence intersects a fixed W_n. *We shall prove that q belongs to W_n.* Let p denote the associated point of $P \subset C''$. Now $p \subset \bar{W}_n$ (in C''). For otherwise there is a neighborhood D'' of p, $D'' \cdot W_n = 0$. We have already observed that q must belong to $D = C \cdot D''$ so that for some j, $W_{n_j} \subset D$ and $W_{n_j} \cdot W_n = 0$ which is contrary to assumption. Therefore $p \subset \bar{W}_n \subset \bar{U}_m''$ for that m for which the given W_n is a component of $C \cdot U_m''$. Then $p \subset U_m''$, since $B(U_m'') \cdot P = 0$ by construction.§ Now if we consider the sets U_i'' which correspond to the W_{n_j}, it is clear that there must occur among them sets U_i'' of indefinitely large subscript, and therefore of arbitrarily small diameter since the U_i'' form a null-sequence. Otherwise it would follow that there were only a finite number of distinct W_{n_j} and this would imply by the monotonic character of these sets that for some k, $W_{n_k} \subset W_{n_j}$ for every j. But then $p = \prod_{j=1}^{\infty} \bar{W}_{n_j} = \bar{W}_{n_k} \supset W_{n_k}$, although $W_{n_k} \subset C$ and is not vacuous. This is absurd. Then, since $p \subset U_n''$ there is a $W_{n_{j'}}$ such that the corresponding $U_i'' \subset U_m''$. It follows, exactly as above, that $p \subset U_i''$. Now $W_{n_{j'}} \subset U_m'' - P \cdot U_m''$ and is connected. Further

† Compare § 3.7, last remark.

‡ If $x^* \subset C$, $x = x^*$.

§ See § 6.2.

$W_{n_j'} \cdot W_n \neq 0$ and W_n is a component of $U_m'' - P \cdot U_m''$. Therefore $W_{n_j'} \subset W_n$ and therefore q belongs to W_n.

6. 5. Now, to return to the argument of § 6. 3, let us suppose that x^* and y^* are points of $Q \subset C^*$ such that $W^*_m \supset x^*$, $W^*_n \supset y^*$ implies $W^*_m \cdot W^*_n \neq 0$, therefore $W_m \cdot W_n \neq 0$. Let W_{m_i} and W_{n_i}, $(i = 1, 2, \cdots)$, define the ideal points x^* and y^*. Then $W_{m_j} \cdot W_{n_k} \neq 0$ for every j and k. If we keep j fixed but $k = 1, 2, \cdots$, we see from the previous section that almost all the $W_{n_k} \subset W_{m_j}$. Keeping k fixed, but letting $j = 1, 2, \cdots$, we see that almost all the $W_{m_j} \subset W_{n_k}$. Then the two sequences are equivalent and define the same ideal point: i. e. $x^* = y^*$. This concludes the argument that C^* is a Hausdorff space. It is trivial that C^* is completely separable. It is clear, also, that C is topologically contained in C^*, and that every point of Q is a limit point (in the topology of C^*) of points of C. Then $C^* = \bar{C}$ and is connected and every $W^*_n = \bar{W}_n$ which is connected, so that C^* is locally connected; where closure is to be understood in the sense of the topology of C^*. To show that C^* is peanian we merely have to prove that it is compact.

To this end let $x^*_1, x^*_2, \cdots$ be any sequence of points of C^*, and $x_1'', x_2'', \cdots$ the corresponding sequence of not necessarily distinct associated points of C'' (if $x_n'' \subset C$, $x_n'' = x^*_n$). We may suppose that the second sequence converges to a point x'' of C'' (if $x_m'' = x_n''$ for some m and infinitely many n, then $x'' = x_m''$).

i) $x'' \subset C \subset C^*$. Let W^*_n be any neighborhood of $x^* = x''$ in C^*. Then $x^* \subset W_n \subset C$. Since W_n is open in C there is a neighborhood U'' of x^* in C'' such that $C \cdot U'' \subset W_n$. Almost all the $x_m'' \subset U''$. If $x_m'' \subset P$, for some m, x^*_m *belongs* to at least one $W_k \subset C \cdot U'' \subset W_n$. Therefore x^*_m *belongs* to W_n and $x^*_m \subset W^*_n$. If $x_m'' \subset C$, $x^*_m = x_m'' \subset W_n \subset W^*_n$. Then it follows that x^* is a limit point in C^* of the sequence $x^*_1, x^*_2, \cdots$.

ii) $x'' \subset P$. Let U_{n_i}'', $(i = 1, 2, \cdots)$, be a monotonic sequence of the neighborhoods generating C'' such that $\Pi U_{n_i}'' = x''$. By § 6. 7, there is a U_{n_k}'', $x'' \subset U_{n_k}'' \subset U_{n_1}''$ such that the points of $C \cdot U_{n_k}''$ are contained in the sum of a finite number of the components of $C \cdot U_{n_1}''$. Now almost all the $x_m'' \subset U_{n_1}''$. Therefore there is at least one component W_{n_1} of $C \cdot U_n''$ such that infinitely many of the points x^*_m *belong* to W_{n_1} and are contained, therefore, in $W^*_{n_1}$. Then it is possible by an easy "diagonalizing" process to find a subsequence

$$x^*_{i_1}, x^*_{i_2}, \cdots, x^*_{i_n}, \cdots,$$

of our given sequence of points of C^*, and a monotonic sequence of neighborhoods

$$W^*_{j_1}, W^*_{j_2}, \cdots, W^*_{j_n}, \cdots,$$

such that each $W^*_{j_k}$ contains almost all of the points of the sequence and such that each U_{n_i}'' of this paragraph contains almost all of the $W_{j_n} = C \cdot W^*_{j_n}$, $(n = 1, 2, \cdots)$. Then the W_{j_n}, $(n = 1, 2, \cdots)$, form a proper sequence associated with the point x'' of C'' and define an ideal point $x^* \subset C^*$. It is clear that $x^* \subset W^*_{j_n}$, for every n. Now *every* neighborhood W^*_k of x^* must contain at least one W_{j_n} and therefore the corresponding $W^*_{j_n}$. Then, finally, x^* is a limit point in C^* of our given sequence.

Then we have shown that C^* is a peanian space, and that C is topologically contained in it.

6. 6. We shall now consider the point-set $Q \subset C^*$. Let the points of $P \subset C''$ be enumerated in a sequence, $p_1, p_2, \cdots$, and let Q_n be the subset of points of Q associated with p_n, $(n = 1, 2, \cdots)$. Then the argument we have just given above shows that Q_n is closed (in C^*). Therefore $Q = \Sigma Q_n$ is an F_σ-set. This is also an obvious consequence of the known *absolute G_δ-character* of the space C. However, the relation of the sets Q and P is not uninteresting.† Let us now show that Q is totally disconnected. This will follow at once when we have shown that the boundaries of our neighborhoods W^*_n are vacuous relative to Q. Now this is merely a restatement of § 6. 4. For if a point $q \subset B(W^*_n)$, every neighborhood $W^*_{n_j}$ of q contains points of W^*_n. If $q \subset Q$ it is an ideal point of C. If W_{n_j}, $(j = 1, 2, \cdots)$, defines q then $W_{n_j} \cdot W_n \neq 0$ and, by § 6. 4, almost every $W_{n_j} \subset W_n$. Then $q \subset W^*_n$ and $q \not\subset B(W^*_n)$. We shall show, finally, that Q is totally avoidable in C^*. Let D^* be any open connected subset of C^*, and suppose that x and y are points of $C \cdot D^*$ which belong to no connected subset of $D^* - Q \cdot D^* = C \cdot D^*$. Now D^* is a locally compact peanian space ‡ and it is known that there must exist a point q of Q such that $W^*_n - Q \cdot W^*_n = C \cdot W^*_n = W_n$ is not connected for every neighborhood W^*_n of q. But this is absurd since every W_n is a connected subset of C by construction. Now since an open connected subset of C is necessarily arcwise connected, the argument shows also that if xy is any arc of C^*, $x + y \subset C$, then there is another arc xy of C in every neighborhood (in C^*) of the given arc.

6. 7. To finish the proof of our Theorem we must show that C^* is uniquely defined by its relation to C. This includes the statement that C^* is a topological invariant of C. We shall prove somewhat more, namely that *if C_1 and C_2 are homeomorphic semi-peanian spaces, $C^*_1 = C_1 + Q_1$, and $C^*_2 = C_2 + Q_2$ the corresponding compactifications with the properties we*

† See § 7.

‡ It makes a pretty terminological sequence to call such spaces *near-peanian.*

have already established, and $T(C_1) = C_2$ *any homeomorphism carrying* C_1 *into* C_2 *then* T *can be extended to a homeomorphism* T^*, $T^*(C^*_1) = C^*_2$, $T^*(C_1) = T(C_1)$.

By a theorem of Alexandroff it will be sufficient to show that T and its inverse are uniformly continuous, since C_1 and C_2 are dense in C^*_1 and C^*_2. By argument of symmetry, it is sufficient to prove this for T. Now to do this it is merely necessary to prove that if $x_1, x_2, \cdots$, and $x'_1, x'_2, \cdots$, are two sequences of points of C_1 converging to the same point x of Q_1, and $y_n = T(x_n)$, $y'_n = T(x'_n)$, then the sequences $y_1, y_2, \cdots$, and $y'_1, y'_2, \cdots$, converge to the same point y of Q_2. Each of the last two sequences certainly has at least one limit point in C^*_2.

Now if either of these has at least two limit points, or if they do not have the same limit point then we can find a subsequence y_{n_i}, $(i = 1, 2, \cdots)$, converging to a point y and a subsequence y'_{m_i}, $(i = 1, 2, \cdots)$, converging to $y' \neq y$. Let x_{n_i} and x'_{m_i} denote the corresponding sequences in C_1. Since C^*_1 is peanian, it contains arcs $x_{n_i}x'_{m_i}$, $(i = 1, 2, \cdots)$, such that these converge to x, i. e. if $z_i \subset x_{n_i}x'_{m_i}$, then z_i converges to x. Now since Q_1 is totally avoidable, we may suppose without any loss that these arcs belong to C_1.† Let $y_{n_i}y'_{m_i} = T(x_{n_i}x'_{m_i})$. These arcs belong to C_2. There is a subsequence of them which converges to a limiting continuum $K \supset y + y'$ of C^*_2. Since Q_2 is totally disconnected, there is at least one point y^* of K, $y^* \subset C_2$, and there is a sequence of points $y^*_1, y^*_2, \cdots$, converging to y^* such that no two of them belong to the same arc $y_{n_i}y'_{m_i}$. Therefore no two of the corresponding points (under the inverse of T) $x^*_1, x^*_2, \cdots$, belong to the same arc $x_{n_i}x'_{m_i}$ and therefore they converge to x. Then the inverse of T cannot be continuous. This contradiction establishes our argument and brings our proof of Theorem II to a close.

7. Here we shall consider the relation of the subset Q of the uniquely defined C^* associated with a semipeanian C and the countable subset P of a compactification C''. We have seen that if we start with a C'' we arrive at C^* with a resolution of Q into ΣQ_n, where each Q_n is closed, every point of a Q_n is associated ‡ with the same point p_n of $P \subset C''$, and the p_n are distinct for distinct Q_n. Now, conversely, if we consider C^* and write $Q = \Sigma Q_n$, where $Q_m \cdot Q_n = 0$, $m \neq n$, and each Q_n is closed, then each such resolution of Q gives rise to a space C''. This space C'' is simply the *decomposition space* of C^* where each point of C and each set Q_n is regarded as a point. For it is

† See the last remark of § 6. 6.

‡ See the opening sentences of § 6. 6.

clear that C'' is peanian, since it is the continuous image (when it is topologized as customarily) of the peanian C^*, and contains C topologically as an everywhere dense subset.

8. There is a simple converse to Theorem II.

THEOREM. *If C^* is peanian and Q a totally disconnected and totally avoidable F_σ, then $C = C^* - Q$ is semipeanian.*

We have shown that C is an s. C.-space.† It is clear from the definition of total avoidability that C is connected and locally connected. It need hardly be remarked that it is not necessary that Q be totally disconnected in order that C be semipeanian.

9. *The space I_2.* The dimension of C^* cannot exceed that of C by more than one, i. e. $\dim C \leqq \dim C^* \leqq 1 + \dim C$. This is an immediate consequence of the totally disconnected character of $Q = C^* - C$. On the other hand, the dimension of C^* may have the larger value. Thus if C is the space I_2 of irrational points of a Cartesian plane (at least one coördinate irrational) then C^* is a topologic sphere. In this case: $\dim C^* = 2, \dim I_2 = 1$. It is amusing that Theorem II permits a characterization of I_2. It is easy to see that I_2 is 1) semipeanian, 2) nowhere locally compact. It is clear, further, that 3) every simple closed curve J of I_2 separates it and 4) no arc of any J separates I_2. Finally, if we follow Freudenthal ‡ and define "ends" abstractly as any monotonic sequence of open connected sets D_n, $(n = 1, 2, \cdots)$, with compact boundaries, such that $\Pi \bar{D}_n = 0$, then 5) I_2 has an at most countable set of *distinct* "ends," distinct being used in the sense of non-equivalent. Although we shall not prove it here it is not difficult to show that these five properties completely characterize I_2.

10. *Primitive skew curves.* By primitive skew curve we understand either of the two non-planar linear graphs.§

THEOREM. *If C contains no primitive skew curve, then C^* contains none.*

The proof is quite simple. For if K^* is a skew curve of C^* then we can replace each arc of K^* with endpoints in C by an arc of C which lies in an

† Compare § 4.

‡ *Loc. cit.*, p. 695. The *distinct* "ends" coincide with our ideal points Q.

§ See C. Kuratowski, "Sur le probleme des courbes gauches en Topologie," *Fundamenta Mathematicae*, vol. 15 (1930), pp. 271-283.

arbitrary neighborhood of the first.† We can conclude easily that C^* contains a skew curve K'' of exactly the same type as K^* whose vertices, at worst, do not belong to C. It is fairly obvious that if these vertices are of order three we can displace K'' slightly at its vertices and obtain a similar skew curve K in C. If the vertices are of order four we may not be able to "reproduce" K'' in C. None the less it is readily seen that by introducing small arcs of C in the neighborhood of the vertices of K'' we can arrive at a skew curve K of C, which is in general of the first type.‡

The theorem above permits a complete extension to semipeanian spaces of the recent work of S. Claytor.§ This work is a very considerable generalization of a Theorem of Kuratowski ¶ on planar subsets.

11. In large part, it has been the burden of this paper that for quasi-peanian spaces at least, local compactness and semicompactness are very close kin. In this concluding section we shall prove the

THEOREM. *A semipeanian group manifold G has at most two distinct "ends" ‖ in the sense of Freudenthal.*

Let t^* denote any point of $G^* - G$, where G^* is the compactification of G in Theorem II, t_n, $(n = 1, 2, \cdots)$, a sequence of points of G converging to t^*, and g any element of G. Now each element of G gives rise to a translation of G into itself, which is a homeomorphism. This extends to a unique homeomorphism of G^* into itself where the complement of G is invariant, by § 6. 7. We may denote this extended homeomorphism by g. The translated points t_ng must converge to t^*. For, if they did not, we could find a neighborhood D^* of t^* with boundary in G such that $t_ng \not\subset D^*$ held for infinitely many n; by thinning our sequence we may say for all n. Now if γ is any arc of G, from g to the identity of G, the translated arcs $t_n\gamma$ must all have at least one point b_n on $B(D^*) \subset G$, where $b_n = t_na_n$, $a_n \subset \gamma$. We may suppose the

† By the total avoidability of $Q = C^* - C$. See the last remark of § 6. 6.

‡ Compare Mazurkiewicz, "Uber nicht plattbare Kurven," *Fundamenta Mathematicae*, vol. 20 (1833), p. 284.

§ I am advised by Claytor that his paper is to appear in the *Annals of Mathematics* in October of this year. See Abstract No. 158, *Bulletin of the American Mathematical Society*, vol. 39 (1933), p. 357.

¶ See note of this section.

‖ See note to § 9, also "Satz 15," *loc. cit.* Our argument here will differ very slightly in form but hardly at all in essence from that of Freudenthal. We are obliged to make this change since local compactness is required by one of his subsidiary theorems (Satz 13).

$a_n \to a$ † $\subset \gamma$, $b_n \to b \subset B(D^*)$. It follows that the $t_n \to t$, $t = ba^{-1} \subset G$. Since this is contrary to assumption, the t_n being a "divergent sequence" ‡ in G the assertion is proved. Now this shows that *each point* of $G^* - G$ is invariant under the homeomorphism g, where g is any element of G.§ If we now consider the element g as fixed the t_n as a sequence of homeomorphisms of G^*, then the $t_n g$ now denote the successive translations of g, and these converge to t^*, for every sequence t_n converging to t^*. By uniformity arguments, the translated sets $t_n M$ where M is any self compact subset of G, converge to t^*, so that for an arbitrary open $D^* \supset t^*$, there is an n such that $t_n M \subset D^*$.¶

Let us suppose that $G^* - G$ contains as many as three distinct points x^*, y^*, z^*. Let $V^* \supset x^*$, $W^* \supset y^*$ be neighborhoods, $z^* \not\subset V^* + W^*$, $V^* \cdot W^* = 0$, and $M = B(W^*) \subset G$. It is an easy consequence of the avoidability of x^* that there is a neighborhood $U^* \supset x^*$, $V^* \supset U^*$, such that *any* two points of $G^* - V^*$ can be joined by an arc of $G^* - U^*$: in particular, the point z^* and any other. By the preceding paragraph there is at least one element x of G such that $xM \subset U^*$. Now with every subset H^* of G^* there is associated the homeomorphic set $x\{H^*\}$. Since z^* is a fixed point, $x\{G^* - W^*\} \supset z^*$. Therefore, since

$$B(x\{G^* - W^*\}) = x\{B(G^* - W^*)\} \subset x\{B(W^*)\} = xM \subset U^*,$$

it follows that $x\{G^* - W^*\} \supset G^* - V^*$. From this it must follow that $x\{W^*\} \subset V^*$. Now this is impossible since $y^* \subset W^*$, $y^* \not\subset V^*$ and is a fixed point under the homeomorphism x. Therefore $G^* - G$ cannot consist of more than two distinct points. This shows at once that G must be locally compact and completes the proof.

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† Read "converge, as elements of the group manifold G, to": of course, they also converge as points.

‡ Associated with the "end" determined by t^*.

§ *Loc. cit.*, "Satz 12."

¶ *Loc. cit.*, "Satz 11."

ADDITION THEOREMS FOR THE DOUBLY PERIODIC FUNCTIONS OF THE SECOND KIND.

By Walter H. Gage.

1. *Introduction.* In this paper we derive addition theorems for $\phi_{\alpha\beta\gamma}(x, y)$, where

$$\phi_{\alpha\beta\gamma}(x, y) = \frac{\vartheta'_1 \vartheta_\alpha(x+y)}{\vartheta_\beta(x)\vartheta_\gamma(y)},$$

and where ϑ_α $(\alpha = 0, 1, 2, 3)$ are the theta functions of Jacobi.* The formulae obtained are addition theorems, not in the ordinary sense, but according to the definition of Poincaré.†

2. *The fundamental formulae.* From the special case of one of Jacobi's theta identities

$$\vartheta_2\vartheta_1(y+v)\vartheta_3(v+x)\vartheta_0(x+y)$$
$$= \vartheta_3(x+y+v)\vartheta_0(x)\vartheta_2(y)\vartheta_1(v) + \vartheta_0(x+y+v)\vartheta_3(x)\vartheta_1(y)\vartheta_2(v)$$

it follows that

$$\vartheta_2(y)\vartheta_1(v)\phi_{001}(x, y+v) + \vartheta_1(y)\vartheta_2(v)\phi_{331}(x, y+v) = \frac{\vartheta'_1\vartheta_2\vartheta_0(x+v)\vartheta_3(x+y)}{\vartheta_0(x)\vartheta_3(x)}. \tag{1}$$

If, in (1), we interchange y and v we also have

$$\vartheta_1(y)\vartheta_2(v)\phi_{001}(x, y+v) + \vartheta_2(y)\vartheta_1(v)\phi_{331}(x, y+v) = \frac{\vartheta'_1\vartheta_2\vartheta_3(x+v)\vartheta_0(x+y)}{\vartheta_0(x)\vartheta_3(x)}. \tag{2}$$

Solving (1) and (2) for $\phi_{331}(x, y+v)$, and simplifying the result by means of the identity

$$\vartheta_1^2(y)\vartheta_2^2(v) - \vartheta_2^2(y)\vartheta_1^2(v) = \vartheta_2^2\vartheta_1(y+v)\vartheta_1(y-v),$$

we get

$$\phi_{331}(x, y+v) = \frac{\vartheta'_1}{\vartheta_2\phi_{111}(y, v)\phi_{122}(y, -v)}\{\phi_{001}(x, v)\phi_{332}(x, y) - \phi_{332}(x, v)\phi_{001}(x, y)\}. \tag{3}$$

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* For the doubly periodic functions see E. T. Bell "Algebraic Arithmetic" page 88; for the definitions and notation of theta functions see Whittaker and Watson "Modern Analysis" Chap. 21 (4th ed.).

† Poincaré, "Sur une Classe Nouvelle de Transcendantes Uniformes," *Journal de Mathématiques*, Quatrième Série, 1890.

Let us write this briefly as

$$(4)\qquad (331) = K(111, 122)\{(001, 332) - (332, 001)\},$$

where

$$K(111, 122) = \vartheta'_1/\vartheta_2 \cdot \phi_{111}(y, v)\phi_{122}(y, -v).$$

Increasing x by $\pi/2$ gives

$$(5)\qquad (001) = K(111, 122)\{(331, 002) - (002, 331)\}.$$

If we increase x by $\pi\tau/2$ in each of (4) and (5), there results

$$(6)\qquad (221) = K(111, 122)\{(111, 222) - (222, 111)\},$$

$$(7)\qquad (111) = K(111, 122)\{(221, 112) - (112, 221)\},$$

respectively.

The remaining formulae for the sixty triple subscripts $\alpha\beta\gamma$ of $\phi_{\alpha\beta\gamma}(x, y+v)$ can be obtained from (4), (5), (6), (7) by using the relations

$$(8)\qquad \phi_{\alpha\beta\gamma}(x, y+v) = \phi_{\alpha\beta\delta}(x, y+v)\frac{\vartheta_\delta(y+v)}{\vartheta_\gamma(y+v)},$$

$$(9)\qquad \phi_{\alpha\beta\gamma}(x, y+v) = \phi_{\alpha\delta\gamma}(x, y+v)\frac{\vartheta_\delta(x)}{\vartheta_\beta(x)}.$$

For example

$$\begin{aligned}(10)\qquad (323) &= (331)\frac{\vartheta_1(y+v)\vartheta_3(x)}{\vartheta_3(y+v)\vartheta_2(x)}\\ &= K(311, 122)\{(001, 332) - (332, 001)\}\frac{\vartheta_3(x)}{\vartheta_2(x)}\\ &= K(311, 122)\{(001, 322) - (322, 001)\},\end{aligned}$$

and

$$(11)\qquad (010) = K(011, 122)\{(331, 012) - (012, 331)\}.$$

3. *The addition formulae.* It follows readily from (4), (10), (11) that

$$\begin{aligned}(12)\quad \phi_{331}(x+u, y+v) &= K(111, 122)K'(011, 122)K'(311, 122)\\ &\quad\cdot\{\phi_{010}(v, x+u)\phi_{323}(y, x+u) - \phi_{323}(v, x+u)\phi_{010}(y, x+u)\}\\ &= K(111, 122)K'(011, 122)K'(311, 122)\\ &\quad\cdot\{\phi_{322}(x, y)\phi_{021}(x, v)\phi_{010}(u, y)\phi_{313}(u, v)\\ &\quad - \phi_{010}(x, y)\phi_{021}(x, v)\phi_{322}(u, y)\phi_{313}(u, v)\\ &\quad + \phi_{010}(x, y)\phi_{313}(x, v)\phi_{322}(u, y)\phi_{021}(u, v)\\ &\quad - \phi_{322}(x, y)\phi_{313}(x, v)\phi_{010}(u, y)\phi_{021}(u, v)\\ &\quad + \phi_{313}(x, y)\phi_{322}(x, v)\phi_{021}(u, y)\phi_{010}(u, v)\\ &\quad - \phi_{021}(x, y)\phi_{322}(x, v)\phi_{313}(u, y)\phi_{010}(u, v)\\ &\quad + \phi_{021}(x, y)\phi_{010}(x, v)\phi_{313}(u, y)\phi_{322}(u, v)\\ &\quad - \phi_{313}(x, y)\phi_{010}(x, v)\phi_{021}(u, y)\phi_{322}(u, v)\},\end{aligned}$$

where K' is the same as K with y and v replaced by x and u respectively. Notice that since $\phi_{\alpha\beta\gamma}(x+u, y+v)$ is equal to $\phi_{\alpha\gamma\beta}(y+v, x+u)$ we can obtain a formula for $\phi_{313}(x+u, y+v)$ by interchanging x and y and u and v.

The formulae for all sixty-four functions can be found as above. By increasing the variables in turn by $\pi/2$ and $\pi\tau/2$ we can also obtain other formulae for each function.

In § 2 we used a formula of Jacobi's containing the constant factor ϑ_2 and consequently K and K' both contain ϑ_2. If we start with a formula containing ϑ_0 or ϑ_3 we get new sets of addition formulae in which the terms corresponding to K and K' contain ϑ_0 or ϑ_3 respectively.

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A THIRD-ORDER IRREGULAR BOUNDARY VALUE PROBLEM AND THE ASSOCIATED SERIES.*

By LEWIS E. WARD.

Introduction. The objects of this paper are to discuss the characteristic functions defined by the system consisting of the differential equation

$$d^3u/dx^3 + [\rho^3 + r(x)]u = 0 \tag{1}$$

and the boundary conditions

$$\begin{aligned} W_1(u) &\equiv \alpha_{12}u''(0) + \alpha_{11}u'(0) + \alpha_{10}u(0) = 0, \\ W_2(u) &\equiv \alpha_{22}u''(0) + \alpha_{21}u'(0) + \alpha_{20}u(0) \\ &\qquad + \beta_{22}u''(\pi) + \beta_{21}u'(\pi) + \beta_{20}u(\pi) = 0, \\ W_3(u) &\equiv \qquad \alpha_{31}u'(0) + \alpha_{30}u(0) = 0, \end{aligned} \tag{2}$$

and to consider the expansion of arbitrary functions in infinite series of these characteristic functions.

In previous papers † on this type of boundary value problem it has been assumed either that the function $r(x)$ appearing in the differential equation possesses a Maclaurin's development in powers of x^3 and that the α's and β's are specially chosen, or that $r(x) \equiv 0$ and the α's and β's are arbitrary except that a certain determinant of the α's should not vanish. As a consequence of these assumptions it was found that an arbitrary function which is to be expanded in an infinite series of the characteristic functions must be analytic at $x = 0$ and its Maclaurin's expansion must have a special form.

In the present paper we make no restriction on the form of the function $r(x)$, supposing only that it is continuous in the interval $0 \leqq x \leqq \pi$ (and for certain theorems either that $r(x)$ has derivatives of all orders on some interval of which $x = 0$ is an interior point, or even that $r(x)$ is analytic at $x = 0$). The hypothesis imposed on the α's and β's in a previous paper is retained, that is, they shall be real constants such that the determinant D_a of the α's arranged as in equations (2) does not vanish, that the matrix

$$\begin{pmatrix} \alpha_{12} & \alpha_{11} & \alpha_{10} \\ 0 & \alpha_{31} & \alpha_{30} \end{pmatrix}$$

* Presented to the American Mathematical Society, February 25, 1933.

† D. Jackson and J. W. Hopkins, *Transactions of the American Mathematical Society*, vol. 20 (1919), p. 245, *et seq.*, and L. E. Ward, *Transactions of the American Mathematical Society*, vol. 29 (1927), p. 716, *et seq.*, and vol. 34 (1932), p. 417, *et seq.*

is of rank two, and that not all the β's are zero. The removal of restrictions on the function $r(x)$ allows us to offer a proof of the validity of the formal expansion of certain functions not necessarily analytic at $x=0$. Due to this feature the proof has to follow lines somewhat different from those employed previously in irregular boundary value problems.

Part I.

This part of the paper is devoted to a study of the characteristic functions. We first define the three functions *

$$\begin{aligned}\delta_1(t) &= e^{\omega_1 t} + e^{\omega_2 t} + e^{\omega_3 t},\\ \delta_2(t) &= e^{\omega_1 t} - \omega_3 e^{\omega_2 t} - \omega_2 e^{\omega_3 t},\\ \delta_3(t) &= e^{\omega_1 t} - \omega_2 e^{\omega_2 t} - \omega_3 e^{\omega_3 t},\end{aligned}$$

in which $\omega_1 = -1$, $\omega_2 = e^{\pi i/3}$, and $\omega_3 = e^{-\pi i/3}$.

Theorem I. *A necessary and sufficient condition that $u(x,\rho)$ satisfy equation* (1) *and the first and third of equations* (2) *is that*

$$\begin{aligned}u(x,\rho) = k[\alpha_{12}\alpha_{31}\rho^2\delta_1(\rho x) + \alpha_{12}\alpha_{30}\rho\delta_2(\rho x) + (\alpha_{11}\alpha_{30} - \alpha_{10}\alpha_{31})\delta_3(\rho x)]\\ - (1/3\rho^2)\int_0^x r(t)\delta_3[\rho(x-t)]u(t,\rho)\,dt,\end{aligned}$$

where k is independent of x.†

To prove the sufficiency we differentiate with respect to x three times both sides of the integral equation in the statement of the theorem. This is seen to result in equation (1). At the same time we verify that the first and third of equations (2) are satisfied.

To prove the necessity we will show that if $u(x,\rho)$ satisfies equation (1), the first and third of equations (2), and also $\alpha_2 u''(0) + \alpha_1 u'(0) + \alpha_0 u(0) = l$, where $l \neq 0$ is given, and $\alpha_2, \alpha_1, \alpha_0$ are chosen so that the determinant

$$D = \begin{vmatrix} \alpha_{12} & \alpha_{11} & \alpha_{10}\\ 0 & \alpha_{31} & \alpha_{30}\\ \alpha_2 & \alpha_1 & \alpha_0 \end{vmatrix}$$

does not vanish, then a value of k, independent of x, exists such that $u(x,\rho)$

* These functions were studied by L. Olivier, *Crelle*, Bd. 2, p. 243. Some of their properties will be found in my 1927 paper, p. 720, already referred to.

† We are concerned only with the solution of equation (1) which is continuous at $x=0$, or if $r(x)$ is analytic at $x=0$, with the solution which is analytic at this point.

In *Comptes Rendus*, t. 90 (1880), p. 721, Y. Villarceau gives the solution of the equation $u^{(m)} \mp rmu = V(x)$. The integral equation of this theorem may be regarded as a special case of Villarceau's formula.

satisfies the integral equation. First we note that α_2, α_1, α_0 can be found such that D does not vanish. Hence a unique $u(x, \rho)$ is determined, which depends upon l. On choosing $k = l/(3D\rho^2)$, it is easy to see that the unique solution $\bar{u}(x, \rho)$ of the integral equation satisfies equation (1), the first and third of equations (2), and $\alpha_2 u''(0) + \alpha_1 u'(0) + \alpha_0 u(0) = l$. Hence $\bar{u}(x, \rho) \equiv u(x, \rho)$.

Because of the homogeneous character of equations (1) and (2) we take $k = 1$ without any loss of generality. Instead of obtaining properties of $u(x, \rho)$ from the above integral equation it is desirable to obtain properties of the solution of

$$(3) \quad u(x, \xi, \rho) = U(x, \xi, \rho) - (1/3\rho^2) \int_\xi^x r(t) \delta_3[\rho(x-t)] u(t, \xi, \rho)\, dt,$$

where
$$U(x, \xi, \rho) \equiv \alpha_{12}\alpha_{31}\rho^2 \delta_1[\rho(x-\xi)] + \alpha_{12}\alpha_{30}\rho\delta_2[\rho(x-\xi)] + (\alpha_{11}\alpha_{30} - \alpha_{10}\alpha_{31})\delta_3[\rho(x-\xi)],$$

since the function defined by this integral equation enters in a later part of the paper. We note that $u(x, \rho) \equiv u(x, 0, \rho)$.

Let m be the exponent of the highest power of ρ with non-zero coefficient in $U(x, \xi, \rho)$, and denote by S_1 the sector of the ρ-plane defined by $0 \leqq \arg \rho \leqq \pi/3$. We prove

THEOREM II. *If $0 \leqq \xi \leqq x \leqq \pi$, and if ρ is in S_1 with $|\rho|$ large, then*

$$u(x, \xi, \rho) = U(x, \xi, \rho) + e^{\omega_3\rho(x-\xi)}\rho^{m-2}E(x, \xi, \rho),^*$$
$$u_x'(x, \xi, \rho) = U_x'(x, \xi, \rho) + e^{\omega_3\rho(x-\xi)}\rho^{m-1}E(x, \xi, \rho),$$
$$u_x''(x, \xi, \rho) = U_x''(x, \xi, \rho) + e^{\omega_3\rho(x-\xi)}\rho^{m}E(x, \xi, \rho).$$

If we define $z(x, \xi, \rho)$ by the equation

$$u(x, \xi, \rho) = U(x, \xi, \rho) + e^{\omega_3\rho(x-\xi)} z(x, \xi, \rho),$$

we find that $z(x, \xi, \rho)$ satisfies the equation

$$z(x, \xi, \rho) = -(1/3\rho^2) e^{\omega_3\rho(\xi - x)} \int_\xi^x r(t)\delta_3[\rho(x-t)] U(t, \xi, \rho)\, dt - (1/3\rho^2)\int_\xi^x r(t)\delta_3[\rho(x-t)] e^{\omega_3\rho(t-x)} z(t, \xi, \rho)\, dt.$$

If M denotes the maximum of $|z(x, \xi, \rho)|$ for $0 \leqq \xi \leqq x \leqq \pi$, we have, for the values of x and ξ which give $|z(x, \xi, \rho)|$ this maximum

* Throughout this paper we denote by E a function of the indicated variables which is bounded when $|\rho|$ is large. Consequently many different bounded functions will be denoted by the same symbol, but no confusion will arise.

$$M \leqq R \mid 3\rho^2 \mid^{-1} \mid e^{\omega_3\rho(\xi-x)} \mid \int_\xi^x \mid \delta_3[\rho(x-t)]U(t,\xi,\rho) \mid dt$$
$$+ RM \mid 3\rho^2 \mid^{-1} \int_\xi^x \mid \delta_3[\rho(x-t)]e^{\omega_3\rho(t-x)} \mid dt,$$

where $R = \max \mid r(t) \mid$ on the interval $0 \leqq t \leqq \pi$.

But on S_1 we have $\mid \delta_n[\rho(x-t)] \mid \leqq 3 \mid e^{\omega_3\rho(x-t)} \mid$, $n = 1, 2, 3$, and $\mid U(t,\xi,\rho) \mid \leqq A \mid \rho^m e^{\omega_3\rho(t-\xi)} \mid$, where A is independent of t, ξ, and ρ. Also $\mid \delta_3[\rho(x-t)]e^{\omega_3\rho(t-x)} \mid \leqq 3$. Hence $M \leqq RA\pi \mid \rho \mid^{m-2} + RM\pi \mid \rho \mid^{-2}$. Hence $M \leqq B \mid \rho \mid^{m-2}$, where B is independent of x, ξ, and ρ. Hence $z(x,\xi,\rho) = \rho^{m-2}E(x,\xi,\rho)$. This gives the first conclusion stated in the theorem.

Now $\displaystyle u'_x(x,\xi,\rho) = U'_x(x,\xi,\rho) + (1/3\rho)\int_\xi^x r(t)\delta_2[\rho(x-t)]u(t,\xi,\rho)dt.$

Hence $\displaystyle \mid u'_x(x,\xi,\rho) - U'_x(x,\xi,\rho) \mid \leqq R \mid 3\rho \mid^{-1} \int_\xi^x \mid \delta_2[\rho(x-t)]u(t,\xi,\rho) \mid dt.$

On putting into this integrand the expression found above for $u(t,\xi,\rho)$ and using inequalities similar to those above, we obtain

$$\mid u'_x(x,\xi,\rho) - U'_x(x,\xi,\rho) \mid \leqq C \mid \rho^{m-1}e^{\omega_3\rho(x-\xi)} \mid,$$

where C is independent of x, ξ, ρ, and from this follows the second conclusion stated in the theorem. The final conclusion is obtained in the same way.

The function $u(x,\xi,\rho)$ is analytic in ρ for every finite ρ, and real when x, ξ, and ρ are real. Hence its Maclaurin's expansion in ρ has real coefficients. Hence, denoting conjugates by dashes, $u(x,\xi,\bar{\rho}) = \overline{u(x,\xi,\rho)}$. This fact will be used in the discussion of the characteristic numbers, and also in the third part of the paper.

The characteristic equation. The characteristic equation is $\Delta(\rho) = 0$, where

$$\Delta(\rho) \equiv \begin{vmatrix} W_1(u_1) & W_1(u_2) & W_1(u_3) \\ W_2(u_1) & W_2(u_2) & W_2(u_3) \\ W_3(u_1) & W_3(u_2) & W_3(u_3) \end{vmatrix},$$

and $u_1(x,\rho)$, $u_2(x,\rho)$, $u_3(x,\rho)$ are any three independent solutions of equation (1). We define $u_i(x,\xi,\rho)$, $i = 1, 2, 3$ by

$$(4)\quad u_i(x,\xi,\rho) = \delta_i[\rho(x-\xi)] - (1/3\rho^2)\int_\xi^x r(t)\delta_3[\rho(x-t)]u_i(t,\xi,\rho)dt, \quad (i = 1, 2, 3).$$

Evidently these three functions, as functions of x, are solutions of equation (1), and

$$u(x,\xi,\rho) = \alpha_{12}\alpha_{31}\rho^2 u_1(x,\xi,\rho) + \alpha_{12}\alpha_{30}\rho u_2(x,\xi,\rho) + (\alpha_{11}\alpha_{30} - \alpha_{10}\alpha_{31}) u_3(x,\xi,\rho).$$

We take $u_i(x,\rho) \equiv u_i(x,0,\rho)$. Then

$$\begin{array}{lll} u_1(0,\rho) = 3 & u_2(0,\rho) = 0 & u_3(0,\rho) = 0 \\ u_1'(0,\rho) = 0 & u_2'(0,\rho) = -3\rho & u_3'(0,\rho) = 0 \\ u_1''(0,\rho) = 0 & u_2''(0,\rho) = 0 & u_3''(0,\rho) = 3\rho^2, \end{array}$$

and

$$\Delta(\rho) = \begin{vmatrix} 3\alpha_{10} & -3\alpha_{11}\rho & 3\alpha_{12}\rho^2 \\ 3\alpha_{20} + W_{2\pi}(u_1) & -3\alpha_{21}\rho + W_{2\pi}(u_2) & 3\alpha_{22}\rho^2 + W_{2\pi}(u_3) \\ 3\alpha_{30} & -3\alpha_{31}\rho & 0 \end{vmatrix},$$

where $\quad W_{2\pi}(u_i) \equiv \beta_{22}u_i''(\pi,\rho) + \beta_{21}u_i'(\pi,\rho) + \beta_{20}u_i(\pi,\rho).$

On expanding the determinant for $\Delta(\rho)$ we obtain

$$\Delta(\rho) = 27D_a\rho^3 - 9\rho W_{2\pi}(u).$$

If we let β_{2j} be that β not equal to zero with the highest second subscript, and use the expressions given in Theorem II for $u(x,\rho)$ and its derivatives, we have

$$\Delta(\rho) = 27D_a\rho^3 + \rho^{m+j+1}e^{\omega_3\rho\pi}[A\delta_k(\rho\pi)e^{-\omega_3\rho\pi} + \rho^{-1}E(\rho)],$$

where A is independent of ρ and is not zero, and k is one of the numbers 1, 2, 3. Hence

$$\Delta(\rho) = \rho^{m+j+1}e^{\omega_3\rho\pi}[A\delta_k(\rho\pi)e^{-\omega_3\rho\pi} + \rho^{-1}E(\rho)].$$

This form is valid if ρ is in the sector S_1 and $|\rho|$ is large.

For $|\rho|$ large the function $\delta_k(\rho\pi)e^{-\omega_3\rho\pi}$ is known to have zeros ρ'_n which are simple and real, with successive zeros separated from one another by a distance which is uniformly bounded from zero. Furthermore, if we construct small circles all of the same radius, centered at the points ρ'_n, and call S'_1 the part of S_1 not inside these circles, we have in S'_1 $|\delta_k(\rho\pi)e^{-\omega_3\rho\pi}| > \delta$, where δ is independent of ρ and is positive.* Hence for $|\rho|$ sufficiently large and ρ in S'_1 we have

$$(5) \qquad |\Delta(\rho)| > h\,|\rho^{m+j+1}e^{\omega_3\rho\pi}|,$$

where h is independent of ρ.

We denote by S_2 and S'_2 the reflections of S_1 and S'_1 in the axis of reals. Then, since $\Delta(\rho)$ takes on in S_2 values conjugate to those it has in S_1, we

* Ward, *loc. cit.*, 1927, pp. 718 and 719.

have in S'_2 for $|\rho|$ large $\Delta(\rho) > h\,|\rho^{m+j+1}e^{\omega_2\rho\pi}|$. Hence for $|\rho|$ large the zeros of $\Delta(\rho)$ can occur only in the small circles. That there is just one in each such small circle and that it is real is shown in the usual way.* These zeros are the characteristic numbers, and are denoted in succession by $\rho_1, \rho_2, \cdots$.

The characteristic functions. The $U(x, \xi, \rho)$ of Theorem II is identical with the $u(x)$ in equation (4) on page 720 of the 1927 paper if a is replaced by ξ. Hence by formula (5) of that paper

$$\begin{aligned} u(x, \xi, \rho) = {} & e^{-\rho(x-\xi)}[\alpha_{12}\alpha_{31}\rho^2 + \alpha_{12}\alpha_{30}\rho + (\alpha_{11}\alpha_{30} - \alpha_{10}\alpha_{31})] \\ & + 2e^{\rho(x-\xi)/2}[\alpha_{12}\alpha_{31}\rho^2 \cos\{3^{1/2}\rho(x-\xi)/2\} \\ & \quad - \alpha_{12}\alpha_{30}\rho \cos\{-\pi/3 + 3^{1/2}\rho(x-\xi)/2\} \\ & \quad - (\alpha_{11}\alpha_{30} - \alpha_{10}\alpha_{31}) \cos\{\pi/3 + 3^{1/2}\rho(x-\xi)/2\}] \\ & + e^{\omega_2\rho(x-\xi)}\rho^{m-2}E(x, \xi, \rho). \end{aligned}$$

On putting $\xi = 0$, and $\rho = \rho_k$, we obtain the characteristic functions of the present paper in the form

$$(6) \quad u_k(x) = 2e^{\rho_k x/2} \begin{Bmatrix} \alpha_{12}\alpha_{31}\rho_k^2 \cos(3^{1/2}\rho_k x/2) - \alpha_{12}\alpha_{30}\rho_k \cos(-\pi/3 + 3^{1/2}\rho_k x/2) \\ - (\alpha_{11}\alpha_{30} - \alpha_{10}\alpha_{31})\cos(\pi/3 + 3^{1/2}\rho_k x/2) \\ + e^{-3\rho_k x/2}\{\alpha_{12}\alpha_{31}\rho_k^2 + \alpha_{12}\alpha_{30}\rho_k + (\alpha_{11}\alpha_{30} - \alpha_{10}\alpha_{31})\}/2 \\ + \rho_k^{m-2}E(x, \rho_k) \end{Bmatrix}.$$

Since ρ_k is real, at least when k is sufficiently large, this form shows clearly the dominant terms in $u_k(x)$.

Part II.

We consider now infinite series of the above characteristic functions,

$$(7) \qquad \sum_{k=1}^{\infty} a_k u_k(x),$$

where the a's are independent of x, and we shall derive certain properties of the sum of such a series. We prove first

Theorem III. *If series (7) converges uniformly in $0 \leqq \alpha \leqq x \leqq x_0 \leqq \pi$, where $\alpha < x_0$, and x_1 is any number less than x_0, then $|a_k| < \gamma\rho_k^{-m}e^{-\rho_k x_1/2}$, where γ is independent of k.*

If k is sufficiently large, we can find a number x'_k in (x_1, x_0) such that any one of the cosines in equation (6) has the value unity for $x = x'_k$. Hence

* Ward, *loc. cit.*, 1932, p. 420.

$|u_k(x'_k)| > \gamma' \rho_k^m e^{\rho_k x'_k/2}$, where γ' is independent of k. But $|a_k u_k(x)| < \gamma''$, where γ'' is independent of x and of k. Hence $|a_k| < \gamma \rho_k^{-m} e^{-\rho_k x'_k/2} < \gamma \rho_k^{-m} e^{-\rho_k x_1/2}$. This inequality can be extended to include all values of k by choosing a different γ if necessary.

THEOREM IV. *If $r(x)$ has derivatives of all orders on the interval $-x_0/2 \leqq x \leqq x_0$, and if the hypothesis of Theorem* III *is satisfied, then the sum $f(x)$ of series* (7) *possesses continuous derivatives of all orders in the interval $-x_2/2 \leqq x \leqq x_2$, where $0 < x_2 < x_1$.*

It is clear from equation (3) and the equations obtained from it by successive differentiations with respect to x that, since $r(x)$ has derivatives of all orders in the interval $-x_0/2 \leqq x \leqq x_0$, the functions $u_k(x)$ will also have derivatives of all orders on this interval, and these derivatives will all be continuous. Also, successive repetitions with slight variations of the argument of Theorem II show that $|u_k^{(j)}(x)| < L_j \rho_k^{m+j} e^{\rho_k x/2}$ if $x \geqq 0$, and $|u_k^{(j)}(x)| < L_j \rho_k^{m+j} e^{-\rho_k x}$ if $x \leqq 0$, where L_j is independent of k and of x. Hence, if x is in the interval $-x_2/2 \leqq x \leqq x_2$, we have $|a_k u_k^{(j)}(x)| < \gamma L_j \rho_k^{j} e^{\rho_k (x_2 - x_1)/2}$. But for each j this is the general term of a convergent series of positive constants, and the series $\Sigma a_k u_k^{(j)}(x)$ converges uniformly in the interval $-x_2/2 \leqq x \leqq x_2$, j being any positive integer or zero. From this follows the conclusion stated in the theorem.

Let us define the w's by means of the equations

$$w_0(x) = f(x), \quad w_n(x) = w'''_{n-1}(x) + r(x) w_{n-1}(x), \quad (n = 1, 2, 3, \cdots).$$

Then $w_n(x) = (-1)^n \sum_{k=1}^{\infty} a_k \rho_k^{3n} u_k(x)$. Hence by the first and third of equations (2)

$$(8) \qquad \left.\begin{aligned} \alpha_{12} w''_n(0) + \alpha_{11} w'_n(0) + \alpha_{10} w_n(0) &= 0 \\ \alpha_{31} w'_n(0) + \alpha_{30} w_n(0) &= 0 \end{aligned}\right\} \qquad (n = 0, 1, 2, \cdots).$$

We have, therefore,

THEOREM V. *If $r(x)$ has derivatives of all orders in an interval of which $x = 0$ is an interior point, and if the hypothesis of Theorem* III *is satisfied, then the sum $f(x)$ of series* (7) *possesses derivatives of all orders at $x = 0$, which satisfy the infinite set of equations* (8).

Equations (8) consist of an infinite set of linear homogeneous equations connecting the values of the derivatives of $f(x)$ at $x = 0$. If they be grouped in pairs, the first pair arising from $n = 0$, the second from $n = 1$, etc., it is

evident from the first pair that one of $f(0)$, $f'(0)$, $f''(0)$ can be chosen arbitrarily, from the second pair that the corresponding one of $f'''(0)$, $f^{\mathrm{IV}}(0)$, $f^{\mathrm{V}}(0)$ can be chosen arbitrarily, and so on. The remaining derivatives then have unique values.

This indicates the degree of arbitrariness in $f(x)$. However, some further restriction beyond equations (8) must be made in order to establish the convergence to $f(x)$ of the formal series. The particular restriction made in this paper is not a necessary condition on $f(x)$, and its statement will be postponed to Part III.

In order to discuss the convergence of series (7) for complex values of x it is desirable to have the asymptotic forms of $u_k(x)$ for large k and for x in certain regions to be defined presently. In order to obtain these forms we shall use equation (3) with $\xi = 0$, allowing x to be a complex variable and ρ a positive constant, and we shall suppose $r(x)$ to be analytic at $x = 0$. We shall take the t-integration over a single straight line. The existence of a unique solution of (3) analytic in x provided x is inside the region containing $x = 0$ in which $r(x)$ is analytic can be shown in the usual way.* We now prove

THEOREM VI. *If $r(x)$ is analytic at $x = 0$ and if T_3 is the finite part of the sector $0 \leqq \arg x \leqq 2\pi/3$, including the boundaries, cut off by a straight line drawn so that T_3 contains no singularity of $r(x)$, then in T_3 we have $u(x,\rho) = U(x,0,\rho) + e^{\omega_3\rho x}\rho^{m-2}E(x,\rho)$, where $E(x,\rho)$ is bounded and analytic in x for ρ large and positive. If T_2 and T_1 are regions similarly constructed in the sectors $4\pi/3 \leqq \arg x \leqq 2\pi$ and $2\pi/3 \leqq \arg x \leqq 4\pi/3$ respectively, then*

$$u(x,\rho) = U(x,0,\rho) + e^{\omega_2\rho x}\rho^{m-2}E(x,\rho) \quad \textit{in } T_2, \textit{ and}$$
$$u(x,\rho) = U(x,0,\rho) + e^{\omega_1\rho x}\rho^{m-2}E(x,\rho) \quad \textit{in } T_1.$$

To give the proof for the region T_3 we write $u(x,\rho) = U(x,0,\rho) + e^{\omega_3\rho x}\rho^{m-2}z(x,\rho)$. From equation (3) we see that $z(x,\rho)$ will satisfy the integral equation

$$z(x,\rho) = -\rho^{-m}\int_0^x r(t)\delta_3[\rho(x-t)]e^{-\omega_3\rho x}U(t,0,\rho)\,dt$$
$$-(1/3\rho^2)\int_0^x r(t)\delta_3[\rho(x-t)]e^{\omega_3\rho(t-x)}z(t,\rho)\,dt.$$

From its definition it is clear that $z(x,\rho)$ is an analytic function of x in the closed region T_3. Let $|z(x,\rho)|$ attain its maximum M in T_3 for $x = x_3$. Then

* See the 1932 paper, pp. 421 and 422, where the proof is given for a special case.

for $x = x_3$ we have $M \leqq | E_1(\rho) | + M | E_2(\rho)\rho^{-2} |$, whence M is a bounded function of ρ, and $z(x, \rho)$ is a bounded function of x and of ρ.

The proofs for the regions T_2 and T_1 are given in a similar way.

We can now consider the convergence of series (7) for complex values of x. Let T_3, T_2, and T_1 be such that they form an equilateral triangle T_{x_2} whose center is at $x = 0$ and one vertex of which is at the point $x = x_2$ on the positive axis of reals.* By Theorem VI we have in T_{x_2} $| u(x, \rho) | \leqq c\rho^m e^{\rho x_2/2}$, where c is independent of x and of ρ. If we suppose the hypothesis of Theorem III is satisfied, then $| a_k u_k(x) | < c\gamma e^{\rho_k(x_2 - x_1)/2}$. If we now take $0 < x_2 < x_1$, the last expression is the general term of a convergent series of positive constants, and series (7) converges uniformly in the interior and on the boundary of T_{x_2}. We have, therefore,

THEOREM VII. *If $r(x)$ is analytic at $x = 0$ and if the hypothesis of Theorem* III *is satisfied, then series* (7) *converges uniformly in the interior and on the boundary of an equilateral triangle T_{x_2} centered at $x = 0$ and having one vertex at $x = x_2$ on the axis of reals between $x = 0$ and $x = x_0$, provided T_{x_2} does not have in its interior or on its boundary a singularity of $r(x)$.*

THEOREM VIII. *If X is the upper limit of all possible choices of the x_0 of Theorem* III, *if $y > X$, and if $r(x)$ has no singularity inside T_y, then series* (7) *cannot converge at any point outside T_X but inside T_y except possibly points on the rays* $\arg x = 0, 2\pi/3, 4\pi/3$.

We omit the proof, which follows the same lines as the proof of Theorem VII, page 423 of the 1932 paper.†

The derivation of equations (8) satisfied by the analytic sum $f(x)$ of series (7) is the same as in the case where the mere existence of all derivatives of $f(x)$ and of $r(x)$ was known. Accordingly we have

THEOREM IX. *If $r(x)$ is analytic at $x = 0$ and if the hypothesis of Theorem* III *is satisfied, then series* (7) *converges to a function $f(x)$ analytic at $x = 0$ and satisfying equations* (8).

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* By the notation T_a we shall mean an equilateral triangle centered at $x = 0$ with one vertex at $x = a$, $a > 0$.

† In the proof there given the point x'_2 is supposed to be such that $0 < \arg x'_2 < 2\pi/3$ instead of $0 \leqq \arg x'_2 \leqq 2\pi/3$, as was incorrectly stated.

PART III.

By the formal series for $f(x)$ we mean a series of type (7) in which the a's are determined by means of certain orthogonality relations involving the adjoint characteristic functions.* It is known that the sum of the first n terms of the formal series for $f(x)$ equals the contour integral

$$(1/2\pi i)\int_{\gamma_n}\int_0^{\pi} 3\rho^2 f(t)G(x,t,\rho)\,dt\,d\rho,\dagger$$

where $G(x,t,\rho)$ is the Green's function of the system (1) and (2), and γ_n is the arc of a circle centered at $\rho=0$, of radius between ρ_n and ρ_{n+1}, and extending from the ray $\arg\rho=-\pi/3$ to the ray $\arg\rho=\pi/3$.

A formula for $G(x,t,\rho)$ useful in the present case is given on page 723 of the 1927 paper. The function $g(x,t,\rho)$ there defined is given by

$$g(x,t,\rho)=\pm(1/2)\sum_{j=1}^{3}u_j(x)v_j(t),\quad +\text{ if } x>t,\quad -\text{ if } x<t,$$

where the u's are any three independent solutions of equation (1), and $v_j(t)$ is the cofactor of $u''_j(t)$ in the determinant

$$W=\begin{vmatrix} u''_1(t) & u''_2(t) & u''_3(t)\\ u'_1(t) & u'_2(t) & u'_3(t)\\ u_1(t) & u_2(t) & u_3(t)\end{vmatrix}\quad\text{divided by } W.$$

It is easy to show that the function $\phi(x)=3\rho^2\sum_{j=1}^{3}u_j(x)v_j(\xi)$ satisfies the integral equation (4) with $i=3$. Hence

$$g(x,t,\rho)=\pm u_3(x,t,\rho)/(6\rho^2),\quad +\text{ if } x>t,\quad -\text{ if } x<t.$$

The formula for $G(x,t,\rho)$ is $G(x,t,\rho)=-N(x,t,\rho)/\Delta(\rho)$, where

$$N(x,t,\rho)=\begin{vmatrix} u_1(x) & u_2(x) & u_3(x) & g(x,t,\rho)\\ W_1(u_1) & W_1(u_2) & W_1(u_3) & W_1(g)\\ W_2(u_1) & W_2(u_2) & W_2(u_3) & W_2(g)\\ W_3(u_1) & W_3(u_2) & W_3(u_3) & W_3(g)\end{vmatrix}.$$

We note that $\Delta(\rho)$ is the minor of $g(x,t,\rho)$ in $N(x,t,\rho)$.

The Green's function is independent of the manner in which $u_1(x)$, $u_2(x)$,

* See the fundamental paper by Birkhoff, *Transactions of the American Mathematical Society*, vol. 9 (1908), p. 373, *et seq.*

† Birkhoff, *loc. cit.*, p. 379.

and $u_3(x)$ be chosen, so long as they are independent solutions of equation (1). We shall take for them the functions defined by equation (4) for $\xi = 0$. This gives

$$N(x,t,\rho) = \begin{vmatrix} u_1(x) & u_2(x) & u_3(x) & g(x,t,\rho) \\ 3\alpha_{10} & -3\alpha_{11}\rho & 3\alpha_{12}\rho^2 & W_1(g) \\ 3\alpha_{20} + W_{2\pi}(u_1) & -3\alpha_{21}\rho + W_{2\pi}(u_2) & 3\alpha_{22}\rho^2 + W_{2\pi}(u_3) & W_2(g) \\ 3\alpha_{30} & -3\alpha_{31}\rho & 0 & W_3(g) \end{vmatrix}.$$

In order to evaluate this determinant we multiply the elements in the first three columns by $v_1(t)/2$, $v_2(t)/2$, and $v_3(t)/2$ respectively, and add these products to the elements in the fourth column. This gives zeros for the second and fourth elements of the fourth column. On expanding by minors of the elements of the fourth column we obtain

$$N(x,t,\rho) = -\Delta(\rho)[g(x,t,\rho) + u_3(x,t,\rho)/(6\rho^2)] - 18\rho u(x) W_{2\pi}(g).$$

But $W_{2\pi}(g) = W_{2\pi}(u_3)/(6p^2)$. Hence

$$\begin{aligned} G(x,t,\rho) &= u_3(x,t,\rho)/(3\rho^2) + 3u(x)W_{2\pi}(u_3)/[\rho\Delta(\rho)] && \text{if } x > t, \\ &= 3u(x)W_{2\pi}(u_3)/[\rho\Delta(\rho)] && \text{if } x < t. \end{aligned}$$

Denoting by $I_n(x)$ the sum of the first n terms of the formal series for $f(x)$, we now have

$$I_n(x) = \frac{1}{2\pi i}\int_{\gamma_n}\int_0^x f(t)u_3(x,t,\rho)\,dt\,d\rho \\ + \frac{1}{2\pi i}\int_{\gamma_n}\frac{9\rho u(x)}{\Delta(\rho)}\int_0^\pi f(t)W_{2\pi}(u_3)\,dt\,d\rho.$$

We introduce the function $\sigma(x,s) = \int_0^x f(t)u_3(s,t,\rho)\,dt$, which will be useful in transforming the integrands of the ρ-integrals in $I_n(x)$. Concerning this function we have first the following theorem.

THEOREM X. *The function $\sigma(x,s)$ satisfies the integral equation*

$$(9) \quad \sigma(x,s) = \int_0^x f(t)\delta_3[\rho(s-t)]\,dt - \frac{1}{3\rho^2}\int_0^x r(t)\delta_3[\rho(s-t)]\sigma(t,t)\,dt \\ - \frac{1}{3\rho^2}\int_x^s r(t)\delta_3[\rho(s-t)]\sigma(x,t)\,dt.$$

This theorem is a restatement of Theorem X of the 1932 paper.

If we put $s = x$, we obtain from equation (9)

$$(10) \quad \sigma(x) = \int_0^x f(t)\delta_3[\rho(x-t)]\,dt - \frac{1}{3\rho^2}\int_0^x r(t)\delta_3[\rho(x-t)]\sigma(t)\,dt,$$

where we have written $\sigma(x) = \sigma(x,x)$.

Before treating the general case it is interesting to consider the special case in which $w_1(x) \equiv 0$. This is the case in which $f(x)$ is a solution of the differential equation $f''' + r(x)f = 0$. We shall suppose that both $f(x)$ and $r(x)$ have derivatives of all orders in the interval $0 \leqq x \leqq \pi$. On integrating by parts three times the first integral in equation (10), that equation becomes

$$\sigma(x) = 3f(x)/\rho - f(0)\delta_1(\rho x)/\rho + f'(0)\delta_2(\rho x)/\rho^2 - f''(0)\delta_3(\rho x)/\rho^3$$
$$-\frac{1}{\rho^3}\int_0^x f'''(t)\delta_3[\rho(x-t)]dt - \frac{1}{3\rho^2}\int_0^x r(t)\delta_3[\rho(x-t)]\sigma(t)\,dt.$$

Now define $\zeta(x)$ by the equation $\sigma(x) = 3f(x)/\rho + \zeta(x)$. Then $\zeta(x)$ satisfies the integral equation

$$\zeta(x) = -f(0)\delta_1(\rho x)/\rho + f'(0)\delta_2(\rho x)/\rho^2 - f''(0)\delta_3(\rho x)/\rho^3$$
$$-\frac{1}{3\rho^2}\int_0^x r(t)\delta_3[\rho(x-t)]\zeta(t)\,dt,$$

in the derivation of which we used the fact that $w_1(x) \equiv 0$. But $\alpha_{12}f''(0) + \alpha_{11}f'(0) + \alpha_{10}f(0) = 0$ and $\alpha_{31}f'(0) + \alpha_{30}f(0) = 0$. Hence $f(0) = \lambda\alpha_{12}\alpha_{31}$, $f'(0) = -\lambda\alpha_{12}\alpha_{30}$, $f''(0) = \lambda(\alpha_{11}\alpha_{30} - \alpha_{10}\alpha_{31})$, where λ is a non-vanishing constant independent of ρ. Hence

$$\zeta(x) = -\lambda U(x, 0, \rho)/\rho^3 - (1/3\rho^2)\int_0^x r(t)\delta_3[\rho(x-t)]\zeta(t)\,dt.$$

On comparing this equation with equation (3) for $\xi = 0$ we infer that $\zeta(x) = -\lambda u(x, 0, \rho)/\rho^3$. Hence we have

$$\sigma(x) = 3f(x)/\rho - \lambda u(x, \rho)/\rho^3. \tag{11}$$

It is in the obtaining of this equation that the necessary conditions (8) enter.

As for the second ρ-integral in $I_n(x)$, we have

$$\int_0^\pi f(t)W_{2\pi}(u_3)\,dt = \beta_{22}\int_0^\pi f(t)u''_3(\pi, t, \rho)\,dt + \beta_{21}\int_0^\pi f(t)u'_3(\pi, t, \rho)\,dt$$
$$+ \beta_{20}\int_0^\pi f(t)u_3(\pi, t, \rho)\,dt,$$

where the accents mean derivatives with respect to the first indicated argument.

Now from $\sigma(x) = \int_0^x f(t)u_3(x, t, \rho)\,dt$ we have $\sigma'(x) = \int_0^x f(t)u'_3(x, t, \rho)\,dt$, since $u_3(x, x, \rho) \equiv 0$. Similarly $\sigma''(x) = \int_0^x f(t)u''_3(x, t, \rho)\,dt$. Hence $\int_0^\pi f(t)W_{2\pi}(u_3)\,dt = 3W_{2\pi}(f)/\rho - \lambda W_{2\pi}(u)/\rho^3$.

We have, therefore,

$$\begin{aligned} I_n(x) &= \frac{1}{2\pi i}\int_{\gamma_n} [3f(x)/\rho - \lambda u(x)/\rho^3 \\ &\qquad + \frac{u(x)}{3D_a\rho^2 - W_{2\pi}(u)}\{3W_{2\pi}(f)/\rho - \lambda W_{2\pi}(u)/\rho^3\}]d\rho \\ &= \frac{1}{2\pi i}\int_{\gamma_n} [3f(x)/\rho + 27u(x)\{W_{2\pi}(f) - \lambda D_a\}/\Delta(\rho)]d\rho. \end{aligned}$$

The cancelling of two large terms in this integrand was due to the form of $\sigma(x)$, which goes back to the form of $f(x)$ imposed in accordance with necessary conditions (8).

On account of the conjugate property of $u(x, \rho)$ in ρ, already referred to, we have

$$I_n(x) = \frac{1}{\pi i}\int_{\gamma'_n} [3f(x)/\rho + 27u(x)\{W_{2\pi}(f) - \lambda D_a\}/\Delta(\rho)]d\rho,$$

where γ'_n is the part of γ_n in S_1. But in S_1 we have $u(x) = \rho^m e^{\omega_3\rho x}E(x, \rho)$, while $W_{2\pi}(f) - \lambda D_a$ is independent of ρ. Recalling inequality (5) we see that $I_n(x) = f(x) + \epsilon_n(x)$, where $\epsilon_n(x)$ tends uniformly to zero as n becomes infinite, x being in the interval $0 \leqq x \leqq \beta < \pi$, where β is any constant between 0 and π. Consequently the formal series for $f(x)$ converges uniformly to $f(x)$ in the interval $0 \leqq x \leqq \beta$. A similar discussion of the convergence of the formal series can be given if $w_k(x)$, $k > 1$, vanishes identically.

If $r(x)$ is analytic at $x = 0$, the uniform convergence of the formal series may be extended to appropriate regions of the x-plane by using Theorems VII and VIII. The largest region of uniform convergence may not be an equilateral triangle. Its shape depends upon the locations of the singularities of $f(x)$ and $r(x)$, and is not discussed here.

In the general case no such simple expression for $\sigma(x)$ as that in equation (11) can be obtained. We shall assume $f(x)$ to possess derivatives of all orders in an interval of which $x = 0$ is an interior point and to possess a continuous second derivative in the interval $0 \leqq x \leqq \pi$. A different form for the integrand of the second ρ-integral in $I_n(x)$ is desirable, and we proceed to the derivation of this. We have

$$\int_0^\pi f(t)W_{2\pi}(u_3)\,dt = \int_0^x f(t)W_{2\pi}(u_3)\,dt + \int_x^\pi f(t)W_{2\pi}(u_3)\,dt.$$

Transforming the first integral on the right in an obvious way results in

$$\int_0^\pi f(t)W_{2\pi}(u_3)\,dt = \beta_{22}\sigma''(x,\pi) + \beta_{21}\sigma'(x,\pi) + \beta_{20}\sigma(x,\pi) + \int_x^\pi f(t)W_{2\pi}(u_3)\,dt,$$

where the accents mean differentiation with respect to s. This gives

$$(12)\quad I_n(x) = -\frac{1}{2\pi i}\int_{\gamma_n}\left[\sigma(x)+\frac{9\rho u(x)}{\Delta(\rho)}\left\{\begin{array}{c}\beta_{22}\sigma''(x,\pi)+\beta_{21}\sigma'(x,\pi)+\beta_{20}\sigma(x,\pi)\\ +\int_x^\pi f(t)W_{2\pi}(u_3)\,dt\end{array}\right\}\right]d\rho.$$

We shall now obtain further properties of the function $\sigma(x, s)$, in which we are interested for $x \leqq s \leqq \pi$. We assume $r(x)$ to possess derivatives of all orders in an interval of which $x = 0$ is an interior point and that the series

$$(13)\qquad \frac{3}{\rho}f(x)-\frac{3}{\rho^4}w_1(x)+\frac{3}{\rho^7}w_2(x)-\cdots$$

converges uniformly in some closed interval J of which $x = 0$ is an interior point.* The latter assumption takes the place of the assumption (made in previous papers) that $f(x)$ be analytic at $x = 0$. It could be lightened, but it is made in this form so that we may have a form of solution of equation (10) to which we can apply equations (8) readily.

Using the defining equations of the w's we see from (13) that the series

$$\frac{3}{\rho}f(x)-\frac{3}{\rho^4}[f'''(x)+r(x)f(x)]+\frac{3}{\rho^7}[w_1'''(x)+r(x)w_1(x)]-\cdots$$

converges uniformly in J, and hence, by subtraction, that

$$\frac{3}{\rho}f'''(x)-\frac{3}{\rho^4}w_1'''(x)+\frac{3}{\rho^7}w_2'''(x)-\cdots$$

converges uniformly in J. Hence, by integration, the series

$$\frac{3}{\rho}[f''(x)-f''(0)]-\frac{3}{\rho^4}[w_1''(x)-w_1''(0)]+\cdots$$

converges unformily in J. But $\frac{3}{\rho}f''(0)-\frac{3}{\rho^4}w_1''(0)+\cdots$ converges, its terms being proportional to those of (13) at $x = 0$ by equations (8). Hence

$$\frac{3}{\rho}f''(x)-\frac{3}{\rho^4}w_1''(x)+\frac{3}{\rho^7}w_2''(x)-\cdots$$

converges uniformly in J, as does also

$$\frac{3}{\rho}f'(x)-\frac{3}{\rho^4}w'_1(x)+\frac{3}{\rho^7}w'_2(x)-\cdots.$$

* The uniform character of the convergence is not necessary for the argument, but is made for convenience.

Consequently, denoting by $\tau(x,\rho)$ the sum of series (13), the x-derivatives of $\tau(x,\rho)$ are obtained by differentiating series (13) termwise.

A set of equations equivalent to (8) is

$$\left.\begin{aligned} w_n(0) &= \lambda_n\alpha_{12}\alpha_{31} \\ w'_n(0) &= -\lambda_n\alpha_{12}\alpha_{30} \\ w''_n(0) &= \lambda_n(\alpha_{11}\alpha_{30}-\alpha_{10}\alpha_{31}) \end{aligned}\right\} \qquad (n=0,1,2,\cdots).$$

These equations serve to define uniquely the λ's, which are independent of ρ. Furthermore the series

$$\lambda_0/\rho-\lambda_1/\rho^4+\lambda_2/\rho^7-\cdots$$

converges, and we denote its sum by $\nu(\rho)$.

THEOREM XI. *If $f(x)$ satisfies equations* (8), *then $\tau(x,\rho)$ satisfies*

$$\begin{aligned} \tau'''+[\rho^3+r(x)]\tau &= 3\rho^2 f(x) \\ \tau(0,\rho) &= 3\alpha_{12}\alpha_{31}\nu(\rho) \\ \tau'(0,\rho) &= -3\alpha_{12}\alpha_{30}\nu(\rho) \\ \tau''(0,\rho) &= 3(\alpha_{11}\alpha_{30}-\alpha_{10}\alpha_{31})\nu(\rho). \end{aligned}$$

These are proved immediately by making use of equations (8) and the series for $\tau(x,\rho)$ and its x-derivatives given above.

THEOREM XII. *If $f(x)$ satisfies equations* (8), *then $\tau(x,\rho)$ satisfies the integral equation*

$$\begin{aligned} \tau(x,\rho) = {} & \frac{\nu(\rho)}{\rho^2}U(x,0,\rho) \\ & +\int_0^x f(t)\delta_3[\rho(x-t)]\,dt-\frac{1}{3\rho^2}\int_0^x r(t)\delta_3[\rho(x-t)]\tau(t,\rho)\,dt. \end{aligned}$$

This is an integral equation equivalent to the differential system in the preceding theorem.

The next theorem gives a form for $\sigma(x)$ analogous to that of the special case treated above.

THEOREM XIII. *If $f(x)$ satisfies equations* (8), *then*

$$\sigma(x)=\tau(x,\rho)-\frac{\nu(\rho)}{\rho^2}u(x).$$

This follows immediately from equation (10), equation (3) with $\xi=0$, and the equation of Theorem 12. We note that the first term in $\nu(\rho)/\rho^2$, namely, λ_0/ρ^3, is the negative of the coefficient of $u(x,\rho)$ in equation (11).

THEOREM XIV. *If $f(x)$ satisfies equations* (8), *then $\sigma(x,s)$ satisfies the integral equation*

$$\sigma(x,s) = \frac{1}{3\rho^2}[\rho^2\delta_1[\rho(s-x)]\tau(x,\rho) - \rho\delta_2[\rho(s-x)]\tau'(x,\rho) + \delta_3[\rho(s-x)]\tau''(x,\rho)]$$
$$- \frac{\nu(\rho)}{\rho^2}u(s,0,\rho) - \frac{\nu(\rho)}{3\rho^4}\int_x^s r(t)\delta_3[\rho(s-t)]u(t,0,\rho)\,dt$$
$$- \frac{1}{3\rho^2}\int_x^s r(t)\delta_3[\rho(s-t)]\sigma(x,t)\,dt.$$

We insert the expression for $\sigma(x)$ obtained in Theorem 13 into equation (9). This gives

$$\sigma(x,s) = \int_0^x f(t)\delta_3[\rho(s-t)]\,dt + \frac{\nu(\rho)}{3\rho^4}\int_0^x r(t)\delta_3[\rho(s-t)]u(t)\,dt$$
$$- \frac{1}{3\rho^2}\int_0^x r(t)\delta_3[\rho(s-t)]\tau(t,\rho)\,dt - \frac{1}{3\rho^2}\int_x^s r(t)\delta_3[\rho(s-t)]\sigma(x,t)\,dt$$

Using Theorem 11 we have for the third integral in this equation

$$\int_0^x r(t)\delta_3[\rho(s-t)]\tau(t,\rho)\,dt$$
$$= \int_0^x \delta_3[\rho(s-t)][3\rho^2 f(t) - \tau'''(t,\rho) - \rho^3\tau(t,\rho)]\,dt$$
$$= 3\rho^2\int_0^x f(t)\delta_3[\rho(s-t)]\,dt - \rho^3\int_0^x \delta_3[\rho(s-t)]\tau(t,\rho)\,dt$$
$$- \int_0^x \delta_3[\rho(s-t)]\tau'''(t,\rho)\,dt.$$

But, integrating by parts three times

$$\int_0^x \delta_3[\rho(s-t)]\tau'''(t,\rho)\,dt = \delta_3[\rho(s-x)]\tau''(x,\rho) - \rho\delta_2[\rho(s-x)]\tau'(x,\rho)$$
$$+ \rho^2\delta_1[\rho(s-x)]\tau(x,\rho) - \delta_3(\rho s)\tau''(0,\rho)$$
$$+ \rho\delta_2(\rho s)\tau'(0,\rho) - \rho^2\delta_1(\rho s)\tau(0,\rho)$$
$$- \rho^3\int_0^x \delta_3[\rho(s-t)]\tau(t,\rho)\,dt.$$

Hence

$$\sigma(x,s) = [\delta_3[\rho(s-x)]\tau''(x,\rho) - \rho\delta_2[\rho(s-x)]\tau'(x,\rho)$$
$$+ \rho^2\delta_1[\rho(s-x)]\tau(x,\rho)]/(3\rho^2)$$
$$- \frac{\nu(\rho)}{\rho^2}\left[U(s,0,\rho) - \frac{1}{3\rho^2}\int_0^x r(t)\delta_3[\rho(s-t)]u(t)\,dt\right]$$
$$- \frac{1}{3\rho^2}\int_x^s r(t)\delta_3[\rho(s-t)]\sigma(x,t)\,dt.$$

On making use of equation (3) with $x = s$ and $\xi = 0$ this becomes the equation of the present theorem.

From the equation of Theorem 14 the desired asymptotic forms of $\sigma(x, s)$ and its s-derivatives can be obtained. Let us write

$$\sigma(x, s) = -\nu(\rho) u(s, 0, \rho)/\rho^2 + \delta_1[\rho(s-x)]f(x)/\rho + e^{\omega_3\rho(s-x)}v(x, s, \rho)/\rho^2.$$

Then $v(x, s, \rho)$ satisfies the equation

$$\begin{aligned} v(x, s, \rho) = e^{\omega_3\rho(x-s)}\,&[\rho^2\delta_1[\rho(s-x)]\{\tau(x, \rho) - 3f(x)/\rho\} \\ &- \rho\delta_2[\rho(s-x)]\tau'(x, \rho) + \delta_3[\rho(s-x)]\tau''(x, \rho)]/3 \\ &- \frac{1}{3\rho^2} f(x) e^{\omega_3\rho(x-s)} \int_x^s r(t)\delta_3[\rho(s-t)]\delta_1[\rho(t-x)]\,dt \\ &- \frac{1}{3\rho^2} e^{\omega_3\rho(x-s)} \int_x^s r(t)\delta_3[\rho(s-t)]e^{\omega_3\rho(t-x)}v(x, t, \rho)\,dt. \end{aligned}$$

Since $\tau(x, \rho)$ and its first two derivatives are continuous in a closed interval, we have $|\tau(x, \rho)|$, $|\tau'(x, \rho)|$, $|\tau''(x, \rho)| < K/|\rho|$, where K is independent of x and of ρ. Also $|\tau(x, \rho) - 3f(x)/\rho| < K/|\rho|^4$, where K has been increased, if necessary. Hence, letting $M(x, \rho)$ be the maximum of $|v(x, s, \rho)|$ for $x \leqq s \leqq \pi$, we have $M(x, \rho) < K' + K''M(x, \rho)/|\rho|^2$, where K' and K'' are both independent of x and of ρ. Here, of course, we have restricted ρ to the sector S_1. It follows that $v(x, s, \rho)$ is an E-function if $|\rho|$ is sufficiently large and ρ is in S_1.

We need also the asymptotic forms of $\sigma'_s(x, s)$ and $\sigma''_s(x, s)$. These are found from the equations obtained from the equation of Theorem 14 by differentiation with respect to s. We incorporate them in the statement of

THEOREM XV. *If equations* (8) *are satisfied, then*

$$\begin{aligned} \sigma(x, s) &= -\nu(\rho)u(s)/\rho^2 + \delta_1[\rho(s-x)]f(x)/\rho + e^{\omega_3\rho(s-x)}\rho^{-2}E(x, s, \rho), \\ \sigma'_s(x, s) &= -\nu(\rho)u'(s)/\rho^2 - \delta_3[\rho(s-x)]f(x) + e^{\omega_3\rho(s-x)}\rho^{-1}E(x, s, \rho), \\ \sigma''_s(x, s) &= -\nu(\rho)u''(s)/\rho^2 + \rho\delta_2[\rho(s-x)]f(x) + e^{\omega_3\rho(s-x)}E(x, s, \rho), \end{aligned}$$

provided $x \leqq s \leqq \pi$, $|\rho|$ *is large, and* ρ *is in* S_1.

We need also the asymptotic form of the t-integral in equation (11). This is given in

THEOREM XVI.

$$\begin{aligned} \int_x^\pi f(t)W_{2\pi}(u_3)\,dt = 3\beta_{20}f(\pi)/\rho + 3\beta_{21}f'(\pi)/\rho + e^{\omega_3\rho(\pi-x)}\rho^{j-2}E(x, \rho) \\ - f(x)[\beta_{22}\rho^2\delta_2[\rho(\pi-x)] - \beta_{21}\rho\delta_3[\rho(\pi-x)] + \beta_{20}\delta_1[\rho(\pi-x)]]/\rho. \end{aligned}$$

This form is obtained by using the special case of Theorem 2 in which

$\alpha_{12}=0$ and $\alpha_{11}\alpha_{30}-\alpha_{10}\alpha_{31}=1$. Taking the asymptotic forms there given, we have

$$\int_x^\pi f(t)W_{2\pi}(u_3)\,dt=\int_x^\pi f(t)\,[\beta_{22}\rho^2\delta_1[\rho(\pi-t)]-\beta_{21}\rho\delta_2[\rho(\pi-t)]$$
$$+\beta_{20}\delta_3[\rho(\pi-t)]]\,dt+\rho^{j-2}\int_x^\pi f(t)\,e^{\omega_3\rho(\pi-t)}E(t,\rho)\,dt.$$

Integrating by parts twice the first integral on the right-hand side of this equation gives an equation equivalent to the one in the statement of the theorem.

We are now ready to insert the results of Theorems 13, 15, and 16 into equation (11). Using at the same time the conjugate property of the integrand in equation (11), we obtain, after making simple reductions,

$$I_n(x)=\frac{1}{\pi i}\int_{\gamma'_n}\left[\begin{array}{l}\tau(x,\rho)-\dfrac{\nu(\rho)u(x)}{\rho^2\Delta(\rho)}\{\Delta(\rho)+9\rho W_{2\pi}(u)\}\\ +\dfrac{27u(x)}{\Delta(\rho)}\{\beta_{20}f(\pi)+\beta_{21}(f'(\pi))\}+\dfrac{u(x)}{\Delta(\rho)}e^{\omega_3\rho(\pi-x)}\rho^{j-1}E(x,\rho)\end{array}\right]d\rho$$

where γ'_n is the part of γ_n in S_1.

But $\tau(x,\rho)=3f(x)/\rho+E(x,\rho)/\rho^4$, $u(x)=\rho^m e^{\omega_3\rho x}E(x,\rho)$, $\Delta(\rho)+9\rho W_{2\pi}(u)=27D_a\rho^3$ and $\nu(\rho)/\rho^2=E(\rho)/\rho^3$. Remembering also inequality (5), we see that $I_n(x)=f(x)+\epsilon_n(x)$, where $\epsilon_n(x)$ tends uniformly to zero as n becomes infinite. We sum this up in

THEOREM XVII. *If*

1) *$f(x)$ and $r(x)$ possess derivatives of all orders in an interval of which $x=0$ is an interior point,*
2) *$r(x)$ and $f''(x)$ are continuous for $0\leqq x\leqq\pi$,*
3) *$f(x)$ satisfies equations* (8), *and*
4) *the series defining $\tau(x,\rho)$ converges uniformly in the interval of hypothesis* 1), *then the formal series for $f(x)$ converges uniformly to $f(x)$ on every closed interval $0\leqq x\leqq\beta<\pi$ interior to the interval mentioned in hypothesis* 1).

UNIVERSITY OF IOWA.

ON THE DIFFERENTIATION OF INFINITE CONVOLUTIONS.

By AUREL WINTNER.

The object of the present note is an elementary theorem on term-by-term differentiation which, when applied to infinite convolutions of distribution functions,† implies results of the following type:

If at least one term $\sigma_k = \sigma_k(x)$, $-\infty < x < +\infty$, *of the convergent infinite convolution* $\sigma_1 * \sigma_2 * \cdots$ *has an absolutely integrable and bounded second derivative, then, as* $n \to \infty$, *the continuous density of* $\sigma_1 * \sigma_2 * \cdots * \sigma_n$ *tends to that of* $\sigma_1 * \sigma_2 * \cdots$ *for every* x.

The assumption that a σ_k has an absolutely integrable and bounded second derivative does not *presuppose* that the Fourier transforms of the densities of the finite and infinite convolutions vanish at infinity more strongly than $o(|t|^{-1})$; and $o(|t|^{-1})$ is an estimate which does not suffice for the absolute integrability of these Fourier transforms.

It will be convenient to consider open intervals only. The classical theorem of Dini on term-by-term differentiation states that if a sequence $\{f_n(x)\}$ of differentiable functions is convergent and the sequence $\{f'_n(x)\}$ is uniformly convergent in an interval (a, b), then $\lim f_n(x)$ is differentiable and its derivative is equal to $\lim f'_n(x)$ at every point of (a, b). This theorem and its usual analogues introduce an assumption regarding the *convergence of the sequence of the derivatives.* For the case of infinite convolutions, a criterion is necessary which is free of such an assumption. A criterion of this type is suggested by, and effectively may be deduced from, the theory of convex functions. It will, however, be convenient to present the proof in a somewhat modified form. One advantage of this presentation is that the proof may easily be extended to the case of more than one variable. The criterion is independent of the Lebesgue theory.

A sequence of functions will be said to be of uniformly bounded variation in (a, b) if the total variation of the n-th function in (a, b) is less than a number which is independent of n. Under this condition the sequence is uniformly bounded in the interval if it is bounded at one point of the interval. The criterion in question runs now as follows:

If a convergent sequence $\{f_n(x)\}$ *of differentiable functions is such that*

† As to terminology, cf. a joint paper of B. Jessen and the present author, appearing in the *Transactions of the American Mathematical Society.*

$\{f'_n(x)\}$ *is uniformly bounded and of uniformly bounded variation in* (a,b), *then*

(i) $\{f_n(x)\}$ *is uniformly convergent in* (a,b);

(ii) $f(x) = \lim f_n(x)$ *has at every point of* (a,b) *a right-hand and a left-hand derivative, and both derivatives are bounded in* (a,b);

(iii) $f'_n(x) \to f'(x)$ *at every* x *for which* $f'(x)$ *exists;*

(iiii) $f'(x)$ *exists with the possible exception of a set of points* x *which is at most enumerable.*

It may be mentioned that $f'_n(x)$ is continuous; in fact, a function of bounded variation cannot have a discontinuity of the second kind and a derivative cannot have a discontinuity of the first kind.

Since $\{f'_n(x)\}$ is uniformly bounded, $\{f_n(x)\}$ satisfies a uniform Lipschitz condition

$$| f_n(x_1) - f_n(x_2) | < M \, | x_1 - x_2 |,$$

where M is independent of x_1, x_2, and n. Now a sequence of functions which satisfy a uniform Lipschitz condition is, according to a theorem of Arzelà, uniformly convergent in (a,b) if it is convergent on a dense set of (a,b). This proves (i). It is seen that it was not necessary to suppose the convergence of $\{f_n(x)\}$ at *every* point of (a,b).

Every uniformly bounded sequence of monotone non-decreasing functions contains an everywhere convergent subsequence; this is a well-known theorem of Helly. It is obvious that if a sequence of functions is uniformly bounded and of uniformly bounded variation, then it may be represented as the difference of two sequences each of which consists of monotone non-decreasing functions which are uniformly bounded. Hence, if a sequence of functions is uniformly bounded and of uniformly bounded variation in (a,b), then it contains a subsequence which is convergent at every point of (a,b).

Let $\{f'_{m_n}(x)\}$ be a convergent subsequence of $\{f'_n(x)\}$. Put $g_n(x) = f_{m_n}(x)$ and let $g'_n(x) \to G(x)$, so that

$$\int_c^x g'_n(t)\,dt \to \int_c^x G(t)\,dt,$$

since $\{g'_n(x)\}$ is uniformly bounded. On the other hand,

$$\int_c^x g'_n(t)\,dt = g_n(x) - g_n(c) \to f(x) - f(c),$$

since $f_n(x) \to f(x)$. Consequently,

$$f(x) - f(c) = \int_c^x G(t)\,dt.$$

This implies (ii) and (iiii), since $G(x)$ is the limit of functions of uniformly bounded variation and is therefore of bounded variation. It is seen that if $f'(x)$ exists at $x = x_0$, then $f'(x_0) = G(x_0)$, so that $g'_n(x_0) \to f'(x_0)$ holds whenever $\{g'_n(x)\}$ is a subsequence of $\{f'_n(x)\}$ which is convergent at every point of (a, b).

Suppose finally that (iii) is false, i. e., that there is a point x_0 such that $f'(x_0)$ exists but $f'_n(x_0) \to f'(x_0)$ does not hold. Since $\{f'_n(x_0)\}$ is a bounded sequence of numbers, it contains a subsequence $\{h'_n(x_0)\}$ such that $h'_n(x_0) \to l$, where $l \neq f'(x_0)$. Consider now the corresponding sequence of functions $\{h'_n(x)\}$; it is a subsequence of $\{f'_n(x)\}$, hence uniformly bounded and of uniformly bounded variation. Thus the sequence $\{h'_n(x)\}$ contains a subsequence $\{g'_n(x)\}$ which is convergent at every point of (a, b). This subsequence of $\{h'_n(x)\}$ is a subsequence of $\{f'_n(x)\}$ and tends therefore at $x = x_0$ to $f'(x_0)$ in virtue of the last remark of the previous paragraph. On the other hand, every subsequence of $\{h'_n(x_0)\}$ tends to l in virtue of $h'_n(x_0) \to l$. Consequently $f'(x_0) = l$. This completes the proof, since $f'(x_0) \neq l$ by hypothesis.

The enumerable set mentioned in (iiii) may actually exist and it may even be dense in (a, b). In fact, it is easy to see that every convex function $f(x)$ satisfying a Lipschitz condition may be approximated by a sequence $\{f_n(x)\}$ of differentiable functions for which $\{f'_n(x)\}$ is uniformly bounded and of uniformly bounded variation. On the other hand, there exist convex functions which satisfy a Lipschitz condition but have a dense set of corners.

The theorem yields a result, viz. (iii), also in cases where the *existence* of the derivative of the limit function is presupposed or obvious for every x. An instance of this situation is the case of infinite convolutions.

Let $\rho(x)$ be a distribution function possessing an absolutely integrable and bounded second derivative. Then $\rho'(x) \leqq C$, where

$$C = \int_{-\infty}^{+\infty} | \rho''(x) | \, dx.$$

Since $\rho'(x)$ and $\rho''(x)$ are bounded Baire functions, they are integrable in the Stieltjes-Lebesgue sense with respect to any distribution function $\tau(x)$. The convolution $\rho * \tau$ has the continuous density

$$\int_{-\infty}^{+\infty} \rho'(x - t) d\tau(t) \qquad (0 \leqq \rho' \leqq C)$$

which is not greater than C, a bound which is independent of x and of the distribution function τ. Furthermore, the total variation of the density of $\rho * \tau$ is

$$\int_{-\infty}^{+\infty} | d\{ \int_{-\infty}^{+\infty} \rho'(x-t)d\tau(t)\}/dx \mid dx = \int_{-\infty}^{+\infty} | \int_{-\infty}^{+\infty} \rho''(x-t)d\tau(t) | \, dx,$$

which is not greater than

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} | \rho''(x-t) | \, dx \, d\tau(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} | \rho''(s) | \, ds \, d\tau(t) = C \cdot 1,$$

where C is independent of the distribution function τ. Hence if σ_1 is a distribution function posssessing an absolutely integrable and bounded second derivative and if $\sigma_2, \sigma_3, \cdots$ are arbitrary distribution functions, it follows, by placing $\rho = \sigma_1$ and $\tau = \sigma_2 * \cdots * \sigma_n$, that the sequence $\{f'_n(x)\}$, where $f_n = \sigma_1 * \cdots * \sigma_n$, exists and is uniformly bounded and of uniformly bounded variation. Finally, if the infinite convolution $\sigma_1 * \sigma_2 * \cdots$ is convergent, then it possesses a continuous density, since $\rho * \tau = \sigma_1 * \tau$ has a continuous density for *any* τ, so that one may choose $\tau = \sigma_2 * \sigma_3 * \cdots$.

It is clear from the proof that the assumption regarding σ_1 may be replaced by a somewhat weaker one, and that higher derivatives of the infinite convolution $\sigma_1 * \sigma_2 * \cdots$ may be similarly treated.

A convergence theory of infinite convolutions has been developed in the joint paper of Jessen and the present author, referred to above. There is an *explicit* sufficient convergence criterion which is of interest insofar as it applies also in cases where the distribution functions occurring in the infinite convolution do not possess finite *second* moments:

If $\sum_{n=1}^{\infty} M_n < +\infty$, *where*

$$M_n = \int_{-\infty}^{+\infty} | x | \, d\sigma_n(x),$$

then the infinite convolution $\sigma_1 * \sigma_2 * \cdots$ *is absolutely convergent.*

In fact, $M_n < +\infty$ implies that the Fourier-Stieltjes transform

$$L(t; \sigma_n) = \int_{-\infty}^{+\infty} e^{itx} d\sigma_n(x)$$

of σ_n has for every t a continuous first derivative of absolute value $\leqq M_n$. Hence $| L(t; \sigma_n) - 1 | \leqq | t | M_n$ in virtue of $L(0; \sigma_n) = 1$. It follows therefore from the convergence of the series $M_1 + M_2 + \cdots$ that the infinite product $L(t; \sigma_1) L(t; \sigma_2) \cdots$ is absolutely and uniformly convergent in every finite t-interval. This means that the infinite convolution $\sigma_1 * \sigma_2 * \cdots$ is absolutely convergent.

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POLYNOMIALS OF BEST APPROXIMATION ASSOCIATED WITH CERTAIN PROBLEMS IN TWO DIMENSIONS.

By W. H. McEwen.

1. *Introduction.* Let $u(x, y)$ be a function which is defined and continuous and possesses continuous partial derivatives of the 1st and 2nd orders throughout a square region of the xy-plane $a \leqq x,\ y \leqq b$. Let C be a closed curve lying wholly within the square, and let J denote the region bounded by C. Then, if it is a question of approximating to $u(x, y)$ throughout the region J by means of polynomials of the form

$$P_{mn}(x, y) = \sum_{i,j}^{m,n} a_{ij} x^i y^j,$$

the problem becomes definite only when a measure of best approximation is determined upon. In this paper we shall consider in turn two different situations as regards the function u and the curve C, designated below as problems A and B respectively, and in each shall define a measure of best approximation and obtain theorems on the convergence of P_{mn} as m, n both become infinite.

Problem A. This problem is characterised by two additional assumptions that we make respecting C and u:

(1) C is an algebraic curve, and hence may be represented by the equation

$$c(x, y) = 0,$$

where $c(x, y)$ is a polynomial of some specified degrees m', n'.

(2) $u(x, y)$ vanishes identically on C, i. e. $u(\alpha, \beta) \equiv 0$, where (α, β) represents a variable point on C.

For the determination of $P_{mn}(x, y)$, the polynomial of best approximation to u of degrees m, n, we shall use

Criterion A. $P_{mn}(x, y)$ must vanish identically on C, $P_{mn}(\alpha, \beta) \equiv 0$, and must give at the same time a minimum value to the expression

$$\iint_J |\nabla^2(u - P_{mn})|^r \, dx\, dy, \qquad \nabla^2 w \equiv \partial^2 w/\partial x^2 + \partial^2 w/\partial y^2,$$

in comparison with all other polynomials of like degrees which vanish identically on C, r being any given constant > 0.

Our special concern will be of course to prove that under suitable addi-

tional hypotheses the polynomials P_{mn} will converge uniformly throughout J to the value of u as m and n both become infinite. Denoting the value of $\nabla^2 u$ by $R(x, y)$, it is clear from the manner in which P_{mn} is defined that the problem could be regarded also from another standpoint, namely that of furnishing an approximation to the solution of a given differential system $\nabla^2 u = R(x, y)$, $u = 0$ on C. However, if this point of view were adopted it would be necessary to introduce into the discussion certain questions of an incidental nature, relating to the extension of the definition of the solution u to apply in that region of the square which lies outside of J. By assuming in the first place that u is defined throughout the square, we have been able to avoid these additional questions and thereby to focus attention more fully upon the processes involved in the proofs of convergence proper.

Problem B. In this case we shall discard the assumptions made in A, so that for the present at least, we may consider C as any closed curve lying wholly within the square, and u as the function described in the first paragraph and taking on arbitrary values on C. For a given pair of positive integers m, n the polynomial of best approximation P_{mn} of degrees m, n will be defined by

Criterion B. P_{mn} must give a minimum value to the expression

$$\iint_J |\nabla^2(u - P_{mn})|^r \, dx\, dy + \lambda \max \text{ on } C \, | u(\alpha, \beta) - P_{mn}(\alpha, \beta)|^s,$$

in comparison with all other polynomials of like degrees, r, s and λ being any given constants > 0, and (α, β) the coördinates of a variable point on C.

For each problem we shall give two proofs of convergence, one based on Hölder's inequality and applicable only when $r > 1$, and the other depending on Markoff's theorem on the derivative of a polynomial in two dimensions and applicable generally when r is any real number > 0. By way of comparison it will be seen that for cases in which $r > 1$ the first method requires a less restrictive hypothesis than the second. The writer has considered already situations in one dimension corresponding to problems A and B,* while Kryloff † has treated a problem similar to A but for approximating sums connected with the method of Ritz.

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* W. H. McEwen, "Problems of closest approximation connected with the solution of linear differential equations," *Transactions of the American Mathematical Society*, vol. 33 (1931), pp. 979-997; "On the approximate solution of linear differential equations with boundary conditions," *Bulletin of the American Mathematical Society*, vol. 38 (1932), pp. 887-894.

† N. Kryloff, "Application de la méthode de l'algorithme variationnel à la solution

In connection with these problems it is of interest to note that the results obtained in this paper, and also the reasoning used with only slight modifications, are valid when the measures of best approximation are altered to the extent that the double integral of the r-th power of $|\nabla^2(u - P_{mn})|$ is replaced by the term

$$\lambda' \max \text{ in } J \mid \nabla^2(u - P_{mn}) \mid^r,$$

r and λ' being any given positive constants. This statement can be made even stronger by asserting that for the case $0 < r \leqq 1$ the hypotheses demanded by our theorems II and IV for convergence can be lightened to agree exactly with that required when $r > 1$.

2. *Preliminary discussion.* In anticipation of later needs we shall develop next some results concerning the simultaneous approximation of an arbitrary function $v(x, y)$ and its partial derivatives of first and second order, by means of polynomials and their corresponding derivatives.

Let $v(x, y)$ be defined throughout the square $a \leqq x,\ y \leqq b$. For simplicity in exposition we shall take this square to be $-1 \leqq x,\ y \leqq 1$, although the results obtained apply equally to the more general case. Suppose further that $v(x, y)$ and its partial derivatives of 1st and 2nd order are continuous throughout the square.

By means of the transformation $x = \cos\theta$, $y = \cos\phi$, we can put v in the form of a periodic function

$$v(\cos\theta, \cos\phi) = \bar{v}(\theta, \phi)$$

having the period 2π in both its arguments θ and ϕ, and thus having the entire $\theta\phi$-plane as its region of definition. Then, by expressing $\bar{v}$ and its derivatives with respect to θ and ϕ in terms of v and its derivatives with respect to x and y, it is readily seen that the hypothesis made in the paragraph above concerning $v(x, y)$ will carry over automatically to $\bar{v}(\theta, \phi)$. Hence for all values of θ and ϕ, $\bar{v}$ and its partial derivatives of 1st and 2nd order are continuous.

But for a periodic function which is continuous, such as $\bar{v}(\theta, \phi)$, Mickelson * has shown that for every pair of positive integers m and n there exists a trigonometric sum of orders m, n

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approchée des équations différentielles aux dérivées partielles du type elliptique," *Bulletin de l'Académie de l'U. R. S. S.*, 1930.

* E. L. Mickelson, "On the approximate representation of a function of two variables," *Transactions of the American Mathematical Society*, vol. 33 (1931), pp. 759-781; p. 76, Theorem II. In this connection see also C. E. Wilder, "On the degree of approximation to discontinuous functions by trigonometric sums," *Rendiconti del*

$$T_{mn}(\theta, \phi) = \sum_{i,j}^{m,n} [A_{ij} \cos i\theta \cos j\phi + B_{ij} \cos i\theta \sin j\phi$$
$$+ C_{ij} \sin i\theta \cos j\phi + D_{ij} \sin i\theta \sin j\phi]$$

such that

$$| \bar{v}(\theta, \phi) - T_{mn}(\theta, \phi) | \leqq K_1 \omega(1/m + 1/n)$$

for all values of θ and ϕ, K_1 being a constant independent of m and n, and $\omega(\delta)$ being the modulus of continuity of $\bar{v}$. It will be well to observe at this point that for functions which are uniformly continuous, such as those with which we will be concerned, $\lim_{\delta \to 0} \omega(\delta) = 0$. The function T_{mn} may be obtained, for example, by making an extension to two dimensions of Jackson's approximating function.* The result is

$$T_{mn}(\theta, \phi) = I_{pq}(\theta, \phi) = h_{pq} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \bar{v}(\theta + 2\lambda, \phi + 2\mu) F_{pq}(\lambda, \mu) \, d\lambda d\mu,$$

where
$$F_{pq}(\theta, \phi) = \left[\frac{(\sin p\lambda)(\sin q\mu)}{(p \sin \lambda)(q \sin \mu)} \right]^4,$$

$$1/h_{pq} = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} F_{pq}(\lambda, \mu) \, d\lambda d\mu,$$

and p and q are two integers such that $2p - 2 \leqq m \leqq 2p$ and $2q - 2 \leqq n \leqq 2q$.

Letting $\xi = \theta + 2\lambda$ and $\eta = \phi + 2\mu$ and substituting under the integral signs for λ, μ, and making use of the fact that the integrand has the period 2π in both the variables ξ and η, we get

$$T_{mn}(\theta, \phi) = I_{pq}(\theta, \phi) = h_{pq} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \bar{v}(\xi, \eta) \Phi(\xi - \theta) \Phi(\eta - \phi) \, d\xi d\eta$$

where
$$\Phi(w) = \tfrac{1}{2}[(\sin pw/2)/(p \sin w/2]^4.$$

On differentiating this result with respect to θ, and replacing $\partial\Phi(\xi - \theta)/\partial\theta$ by its equal $-[\partial\Phi(\xi - \theta)/\partial\xi]$, and then integrating the resulting expression by parts, we get

$$\frac{\partial}{\partial\theta} T_{mn}(\theta, \phi) = h_{pq} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial\xi} \bar{v}(\xi, \eta) \cdot \Phi(\xi - \theta) \Phi(\eta - \phi) \, d\xi d\eta,$$

which is precisely the I_{pq}-function associated with $\partial\bar{v}/\partial\theta$. Then, since $\partial\bar{v}/\partial\theta$ is

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Circolo Matematico di Palermo, vol. 39 (1915), pp. 345-361; p. 358, Theorem X, in which it is shown that if the given function satisfies a Lipschitz condition the absolute value of the error will not exceed a constant multiple of $(1/m + 1/n)$.

* D. Jackson, "The theory of approximations," *American Mathematical Society Colloquium Publications*, New York, 1930, p. 3.

itself continuous, there must exist, according to Mickelson's result, a constant K_2 independent of m and n such that

$$|\partial\bar{v}/\partial\theta - \partial T_{mn}/\partial\theta| \leqq K_2\omega(1/m + 1/n)$$

for all values of θ and ϕ. Similarly we can show that the remaining derivatives of $\bar{v} - T_{mn}$ of 1st and 2nd order have upper bounds which are constant multiples of $\omega(1/m + 1/n)$.

Furthermore, since $\bar{v}(\theta, \phi) = v(\cos\theta, \cos\phi)$ is an even function of both θ and ϕ, the function T_{mn} will necessarily be even. Hence, on changing back again to the variables x, y, we are led at once to a polynomial $p_{mn}(x, y)$ of degrees m, n, while the region of approximation becomes again the square $-1 \leqq x, y \leqq 1$. Moreover from the identities

$$\bar{v}(\theta, \phi) - T_{mn}(\theta, \phi) = v(x, y) - p_{mn}(x, y),$$

$$\frac{\partial}{\partial\theta}[\bar{v}(\theta, \phi) - T_{mn}(\theta, \phi)] = (1 - x^2)^{1/2} \frac{\partial}{\partial x}[v(x, y) - p_{mn}(x, y)],$$

$$\frac{\partial^2}{\partial\theta^2}[\bar{v}(\theta, \phi) - T_{mn}(\theta, \phi)] = (1 - x^2) \frac{\partial^2}{\partial x^2}[v(x, y) - p_{mn}(x, y)] + x\frac{\partial}{\partial x}[v(x, y) - p_{mn}(x, y)], \text{ etc.}$$

it is clear that if we restrict our attention to a region J which lies wholly within the square, so that $(1 - x^2)^{1/2}$ has a positive lower bound, then the quantities $(\partial^{i+j}/\partial x^i\partial y^j)[v(x, y) - p_{mn}(x, y)]$, $(i + j = 0, 1, 2)$, will have upper bounds in J which are constant multiples of $\omega(1/m + 1/n)$.

Furthermore, if the hypothesis regarding v is extended so that v and its partial derivatives of orders $1, 2, \cdots, k$ $(k > 2)$ are continuous throughout the square, then by an appropriate generalization of the function I_{pq},* the argument given above can be used to prove that the expressions

$$\partial^{i+j}(v - p_{mn})/\partial x^i\partial y^j, \qquad (i + j = 0, 1, 2)$$

have upper bounds in J which are constant multiples of

$$(1/m + 1/n)^k\,\Omega(1/m + 1/n),$$

where $\Omega(\delta)$ is the greatest of the moduli of continuity associated with the k-th order derivatives of v.

The results obtained in this section thus far may be summarized in the following two theorems.

* See Mickelson, *loc. cit.*, pp. 766-768.

THEOREM A. *If $v(x, y)$ and its partial derivatives of the 1st and 2nd orders are continuous throughout the square $a \leqq x, y \leqq b$, then for every pair of positive integers m and n there exists a polynomial $p_{mn}(x, y)$ of degrees m, n, and a positive constant K independent of m and n, such that the relations*

$$|\partial^{i+j}(v - p_{mn})/\partial x^i \partial y^j| \leqq K\omega(1/m + 1/n), \qquad (i + j = 0, 1, 2)$$

hold uniformly throughout any closed region J which lies wholly within the square.

THEOREM B. *If $v(x, y)$ and its partial derivatives of orders $1, 2, \cdots, k$ are continuous throughout $a \leqq x, y \leqq b$, then, for every pair of positive integers m, n, there exists a polynomial $p_{mn}(x, y)$ of degrees m, n, and a positive constant K′ independent of m and n, such that the relations*

$$|\partial^{i+j}(v - p_{mn})/\partial x^i \partial y^j| \leqq K'(1/m + 1/n)^k \cdot \Omega(1/m + 1/n) \quad (i + j = 0, 1, 2)$$

hold uniformly throughout the region J.

As yet we have not considered the questions of existence and uniqueness in relation to our polynomials of best approximation. It is not difficult to show that in both problems polynomials P_{mn} as defined by the respective criteria do exist, and moreover when $r > 1$ are uniquely determined. Thus in problem A where C is an algebraic curve represented by $c(x, y) = 0$, and P_{mn} is required to vanish identically on C, we can write

$$P_{mn}(x, y) = \sum_{i,j}^{m-m',n-n'} b_{ij}\psi_{ij}(x, y),$$

where $\psi_{ij}(x, y) = c(x, y)x^i y^j$. Then since no polynomial which so vanishes can be harmonic in J (unless it be identically zero there), it follows that $\nabla^2 P_{mn} = \sum_{i,j}^{m-m',n-n'} b_{ij}\nabla^2\psi_{ij}$ cannot vanish identically in J and hence that it may be regarded as a linear combination of functions $\nabla^2\psi_{ij}$ which are linearly independent in J. On the basis of this result the existence and uniqueness theorems can be proved by the use of an argument exactly similar to that used in the one dimensional problem.* By a suitable modification of the wording the same type of argument would suffice also in problem B.

3. *Problem A. Convergence in the special case* $r > 1$. Consider the function $v(x, y) = u(x, y)/c(x, y)$. Let $p_{m-m',n-n'}$ be a polynomial of degrees $m - m'$, $n - n'$, arbitrary for the moment, and let $\epsilon > 0$ be such that the relations

* See the writer's first paper, *loc. cit.*

$$(1) \qquad | \partial^{i+j}(v - p_{m-m',n-n'})/\partial x^i \partial y^j | \leqq \epsilon \qquad (i+j=0,1,2)$$

hold uniformly throughout J. Ultimately we shall assume that v satisfies the hypothesis of Theorem A, so that ϵ may be taken to be

$$K\omega[1/(m-m') + 1/(n-n')]$$

and hence $\lim_{m,n\to\infty} \epsilon = 0$.

Let $\pi_{mn} = c p_{m-m',n-n'}$. Then π_{mn} is a polynomial of degrees m, n, and furthermore

$$u - \pi_{mn} = c(v - p_{m-m',n-n'}),$$
$$\frac{\partial}{\partial x}(u - \pi_{mn}) = (v - p_{m-m',n-n'})\frac{\partial c}{\partial x} + c\frac{\partial}{\partial x}(v - p_{m-m',n-n'}), \text{ etc.}$$

From these relations and (1) it is clear that the upper bounds in J of $| \partial^{i+j}(u - \pi_{mn})/\partial x^i \partial y^j |$, $(i+j=0,1,2)$ are expressible linearly in terms of ϵ and the upper bounds of c and its derivatives. Hence there must exist a constant B independent of m and n to satisfy the inequalities

$$(2) \qquad | \partial^{i+j}(u - \pi_{mn})/\partial x^i \partial y^j | \leqq B\epsilon \qquad (i+j=0,1,2)$$

uniformly throughout J. In particular then

$$(3) \qquad | \nabla^2(u - \pi_{mn}) | \leqq 2B\epsilon.$$

Now the polynomial of best approximation P_{mn} of degrees m, n is defined so as to vanish on C and at the same time to minimize the expression

$$\iint_J | \nabla^2(u - P_{mn}) |^r \, dxdy$$

in comparison with all other polynomials of like degrees which so vanish. Such another polynomial is π_{mn}. Hence, by virtue of this and (3), we can write

$$(4) \qquad \iint_J | \nabla^2(u - P_{mn}) |^r \, dxdy$$
$$\leqq \iint_J | \nabla^2(u - \pi_{mn}) |^r \, dxdy \leqq A(2B\epsilon)^r,$$

A being the area of the region J.

Let $G(x, y; \xi, \eta)$ be the Green's function of two dimensions associated with the homogeneous differential system $\nabla^2 w = 0$, $w = 0$ on C. Then, since u and P_{mn} both vanish identically on C, it is possible to write

$$u(x,y)=\iint_J G(x,y;\xi,\eta)\nabla^2 u(\xi,\eta)\,d\xi\,d\eta,$$

$$P_{mn}(x,y)=\iint_J G(x,y;\xi,\eta)\nabla^2 P_{mn}(\xi,\eta)\,d\xi\,d\eta,$$

and therefore also

$$u-P_{mn}=\iint_J G(x,y;\xi,\eta)\nabla^2[u(\xi,\eta)-P_{mn}(\xi,\eta)]\,d\xi\,d\eta.$$

The function G is not bounded in J (becoming infinite as

$$\log\sqrt{(x-\xi)^2+(y-\eta)^2}$$

at the point (ξ,η)), but nevertheless the double integrals over J of $|G|$ and $|G|^{r/(r-1)}$ are finite in value. Hence, the number r being >1, it is possible to apply Hölder's inequality to this last relation and so obtain the result

$$|u-P_{mn}|\leqq\Big[\iint_J |G|^{r/(r-1)}d\xi\,d\eta\Big]^{1-1/r}\cdot\Big[\iint_J |\nabla^2(u-P_{mn})|^r\,d\xi\,d\eta\Big]^{1/r}.$$

The first factor occurring on the right is merely a constant, whereas the second is bounded as shown in (4). Hence there must exist a constant D independent of m and n to satisfy the relation

$$|u-P_{mn}|\leqq D\epsilon$$

uniformly throughout J.

If we assume now that $v=u/c$ satisfies the hypothesis of Theorem A so that ϵ may be chosen to make $\lim_{m,n\to\infty}\epsilon=0$, then it is certain from this last result that P_{mn} will converge uniformly in J to the value of u as m and n both become infinite.

Likewise we can show that the partial derivatives of the 1st and 2nd orders converge. For we can write

$$\frac{\partial^{i+j}}{\partial x^i\partial y^j}(u-P_{mn})=\iint_J\frac{\partial^{i+j}G}{\partial x^i\partial y^j}\cdot\nabla^2(u-P_{mn})\,d\xi\,d\eta,\qquad(i+j=1,2),$$

and so by Hölder's inequality and (4) obtain

$$|\partial^{i+j}(u-P_{mn})/\partial x^i\partial y^j|\leqq D'\epsilon,\qquad(i+j=1,2),$$

where D' is a constant independent of m and n. Thus we can state

THEOREM I. *In problem A in the case when $r>1$, if the function*

$u(x,y)/c(x,y)$ satisfies the hypothesis of Theorem A, there will exist a positive quantity $\epsilon_{mn}=[1/(m-m')+1/(n-n')]$, and a positive constant D_1 independent of m and n, such that the relations

$$|\,\partial^{i+j}(u-P_{mn})/\partial x^i\partial y^j\,|\leqq D_1\epsilon_{mn}, \qquad (i+j=0,1,2)$$

hold uniformly throughout J, and $\lim\limits_{m,n\to\infty}\epsilon_{mn}=0$.

4. *Problem A. Convergence in the general case when $r>0$.* Let $F(x,y)=u(x,y)-\pi_{mn}(x,y)$, where π_{mn} is the polynomial described in the last section satisfying relations (2) and (3),

$$(2) \qquad |\,\partial^{i+j}(F)/\partial x^i\partial y^j\,|\leqq B\epsilon, \qquad (i+j=0,1,2),$$

$$(3) \qquad |\,\nabla^2(F)\,|\leqq 2B\epsilon.$$

Then the function F, like u, vanishes identically on C and hence there will exist for it a polynomial of best approximation Q_{mn} of degrees m,n (Criterion A). Moreover Q_{mn} will vanish identically on C and the double integral

$$\gamma=\iint_J |\,\nabla^2(F-Q_{mn})\,|^r\,dx\,dy$$

will be a minimum for polynomials of like degrees which so vanish. But 0 may be regarded as another such polynomial vanishing on C, and hence $\gamma\leqq\iint_J |\,\nabla^2(F)\,|^r\,dx\,dy$, and therefore by reason of (3),

$$(6) \qquad \gamma\leqq A(2B\epsilon)^r.$$

Let δ be the maximum value of $|\,\nabla^2Q_{mn}\,|$ in the region J, and let (x_0,y_0) be a point of J at which $|\,\nabla^2Q_{mn}(x_0,y_0)\,|=\delta$. Then, since ∇^2Q_{mn} is a polynomial of degrees not exceeding m,n, it follows as a consequence of Markoff's theorem that $|\,\partial\nabla^2Q_{mn}/\partial x\,|\leqq H\bar{m}^2\delta$ and $|\,\partial\nabla^2Q_{mn}/\partial y\,|\leqq H\bar{m}^2\delta$ throughout the region J, $\bar{m}$ being the greater of the two numbers m and n, and H a constant depending on the region J * and independent of m and n. In the light of these results and the mean value theorem we can write

$$|\,\nabla^2Q_{mn}(x,y)-\nabla^2Q_{mn}(x_0,y_0)\,|\leqq[\,|\,x-x_0\,|+|\,y-y_0\,|\,]H\bar{m}^2\delta.$$

* In this connection it should be observed that certain broad requirements must be met by the region J in order to insure the applicability of Markoff's theorem. It will be sufficient to assume that J is a region for which there exists a positive constant h and a small angle $\theta\neq 0$ such that from every point of the boundary curve two line segments of lengths h and inclined at an angle θ with one another can be drawn belonging wholly to the region.

Now let us consider the square about the point (x_0, y_0) defined by the inequalities $| x - x_0 | \leqq 1/(4H\bar{m}^2)$, $| y - y_0 | \leqq 1/(4H\bar{m}^2)$. If j represent that part of the square which belongs to J, then throughout j, by virtue of the relation written above,

$$| \nabla^2 Q_{mn}(x, y) - \nabla^2 Q_{mn}(x_0, y_0) | \leqq \delta/2,$$

and hence

$$| \nabla^2 Q_{mn}(x, y) | \geqq \delta/2. \tag{7}$$

Let us assume for the moment that $\epsilon < \delta/(8B)$, so that $| \nabla^2(F) | \leqq 2B\epsilon < \delta/4$. Then, by (7), $| \nabla^2(F - Q_{mn}) | > \delta/4$ throughout j, and hence

$$\gamma = \iint_J | \nabla^2(F - Q_{mn}) |^r \, dx \, dy$$

$$\geqq \iint_j | \nabla^2(F - Q_{mn}) |^r \, dx \, dy > [1/(4Hm^2)^2](\delta/4)^r.$$

Therefore $\delta \leqq 4[16H^2\bar{m}^4\gamma]^{1/r}$, and by (6)

$$\delta \leqq 4[16H^2\bar{m}^4 A]^{1/r}(2B\epsilon).$$

This result was proved on the basis of the assumption $\epsilon < \delta/(8B)$. However is this inequality does not hold, then $\delta \leqq 8B\epsilon$. Hence in any case there will exist a constant E independent of m and n such that

$$\delta \leqq E\bar{m}^{4/r}\epsilon. \tag{8}$$

But the function Q_{mn} may be expressed in terms of the Green's function,

$$Q_{mn}(x, y) = \iint_J G(x, y; \xi, \eta) \nabla^2 Q_{mn}(\xi, \eta) \, d\xi \, d\eta.$$

Hence throughout J

$$| Q_{mn} | \leqq \delta \iint_J | G(x, y; \xi, \eta) | \, d\xi \, d\eta = W\delta,$$

where W is a finite constant. Therefore, by (8),

$$| Q_{mn} | \leqq WE\bar{m}^{4/r}\epsilon.$$

From this and (2) it follows that

$$| F - Q_{mn} | \leqq B\epsilon + WE\bar{m}^{4/r}\epsilon \leqq L\bar{m}^{4/r}\epsilon,$$

where $L = (B + WE)$ is independent of m and n.

Now let us assume that $v = u/c$ satisfies the hypothesis of Theorem B with the integer k taken $\geqq 4/r$, so that ϵ may be given the value $K'[1/(m-m') + 1/(n-n')]^k \Omega[1/(m-m') + 1/(n-n')]$. Then as m, n both become infinite * $\bar{m}^{4/r}\epsilon$ will approach zero as a limit and therefore the quantity $| F - Q_{mn} |$ will converge to zero. But, as we have noted already, $F - Q_{mn}$ is identical with $u - P_{mn}$, where P_{mn} is the polynomial of best approximation to u. Thus we have proved that under the hypotheses stated P_{mn} converges to u. In a like manner it can be shown that the partial derivatives of the 1st and 2nd orders of P_{mn} converge to the respective derivatives of u. The results of this section are set forth in

THEOREM II. *In problem A in the general case $r > 0$, if the curve C is subject to the limitations imposed by the requirements of Markoff's theorem (see footnote *, p. 375), and if the function u/c satisfies the hypothesis of Theorem B with the integer k taken $\geqq 4/r$, then there will exist a positive constant D_2 independent of m and n, and a positive quantity ϵ_{mn} such that the relations*

$$| \partial^{i+j}(u - P_{mn})/\partial x^i \partial y^j | \leqq D_2 \epsilon_{mn}, \qquad (i + j = 0, 1, 2)$$

hold uniformly throughout J, and furthermore, provided m and n maintain the same order of magnitude,

$$\lim_{m,n \to \infty} \epsilon_{mn} = 0.$$

An explicit formula for ϵ_{mn} is

$$\bar{m}^{4/r}[1/(m-m') + 1/(n-n')]^k \Omega[1/(m-m') + 1/(n-n')].$$

5. *Problem B. Convergence in the special case $r > 1$.* From this point on we discard the suppositions made in problem A that C be algebraic and that u vanish identically on C. Let p_{mn} be a polynomial of degrees m, n, which for the moment may be regarded as arbitrary, and let $\epsilon > 0$ satisfy the relations

$$(10) \qquad | \partial^{i+j}(u - p_{mn})/\partial x^i \partial y^j | \leqq \epsilon, \qquad (i + j = 0, 1, 2)$$

uniformly throughout J. Then also

$$(11) \qquad | \nabla^2 (u - p_{mn}) | \leqq 2\epsilon.$$

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* It must be understood here that m, n become infinite in such a way as to maintain at all times the same order of magnitude. That is, there must exist a constant α to satisfy the inequalities $1 \leqq \bar{m}/m \leqq \alpha$, $1 \leqq \bar{m}/n \leqq \alpha$. Then the coefficient of Ω in $\bar{m}^{4/r}\epsilon$ will not exceed $\alpha^{4/r}(m^{4/(kr)}/(m-m') + n^{4/(kr)}/(n-n'))^k$, a quantity which has a finite limit when $k \geqq 4/r$.

But the polynomial of best approximation P_{mn} of degrees m, n (see Criterion B) is now defined to give a minimum value to the expression

$$\gamma = \iint_J \mid \nabla^2(u - P_{mn}) \mid^r dx\, dy + \lambda \max \mid u(\alpha, \beta) - P_{mn}(\alpha, \beta) \mid^s$$

in comparison with all other polynomials of degrees m, n, and therefore in particular with the polynomial p_{mn}. Hence

$$\gamma \leqq \iint_J \mid \nabla^2(u - p_{mn}) \mid^r dx\, dy + \lambda \max \mid u(\alpha, \beta) - p_{mn}(\alpha, \beta) \mid^s,$$

and therefore, by virtue of (10) and (11),

$$\gamma \leqq A(2\epsilon)^r + \lambda\epsilon^s.$$

Ultimately ϵ will be made to approach zero and so at this point we may assume that $2\epsilon < 1$. Then if q denote the smaller of the two numbers r, s

$$\gamma \leqq (A + \lambda)(2\epsilon)^q.$$

But each term of γ is $\geqq 0$ and hence each $\leqq \gamma$, and therefore

$$(12) \qquad \iint_J \mid \nabla^2(u - P_{mn}) \mid^r dx\, dy \leqq (A + \lambda)(2\epsilon)^q,$$

$$(13) \qquad \max \mid u(\alpha, \beta) - P_{mn}(\alpha, \beta) \mid \leqq [(A + \lambda)(2\epsilon)^q/\lambda]^{1/s}.$$

The function u may be expressed in terms of the Green's function,

$$u(x, y) = \iint_J G(x, y; \xi, \eta) \nabla^2 u(\xi, \eta) d\xi\, d\eta + \phi(x, y),$$

where $\phi(x, y)$ is a function which is harmonic in the region J and which, on the boundary C, takes on the same values as does $u(x, y)$, i. e. $\phi(\alpha, \beta) = u(\alpha, \beta)$. So also,

$$P_{mn}(x, y) = \iint_J G(x, y; \xi, \eta) \nabla^2 P_{mn}(\xi, \eta) d\xi\, d\eta + \psi(x, y),$$

where $\psi(x, y)$ is harmonic in J and $\psi(\alpha, \beta) = P_{mn}(\alpha, \beta)$. Then

$$u - P_{mn} = \iint_J G \cdot \nabla^2(u - P_{mn}) d\xi\, d\eta + [\phi(x, y) - \psi(x, y)].$$

The number r being > 1, Hölder's inequality can be applied to the first term on the right to give

$$\left| \iint_J G \cdot \nabla^2(u - P_{mn}) d\xi\, d\eta \right|$$
$$\leqq \left[\iint_J |G|^{r/(r-1)} d\xi\, d\eta \right]^{1-1/r} \cdot \left[\iint_J |\nabla^2(u - P_{mn})|^r d\xi\, d\eta \right]^{1/r},$$

from which it follows, by virtue of (12), that a constant M independent of m and n can be found such that

$$\left| \iint_J G \cdot \nabla^2(u - P_{mn}) d\xi\, d\eta \right| \leqq M\epsilon^{q/r}.$$

On the other hand the second term $(\phi - \psi)$, being harmonic in J, will take on its maximum values on the boundary C, so that

$$|\phi(x, y) - \psi(x, y)| \leqq \max |\phi(\alpha, \beta) - \psi(\alpha, \beta)|.$$

But $\phi(\alpha, \beta) - \psi(\alpha, \beta) = u(\alpha, \beta) - P_{mn}(\alpha, \beta)$ and hence, by (13),

$$|\phi - \psi| \leqq \max |u(\alpha, \beta) - P_{mn}(\alpha, \beta)| \leqq [(A + \lambda)(2\epsilon)^q/\lambda]^{1/s} = N\epsilon^{q/s},$$

where N is a constant independent of m and n. Thus we can write

$$|u - P_{mn}| \leqq M\epsilon^{q/r} + N\epsilon^{q/s} \leqq (M + N)\epsilon,$$

a relation which holds uniformly throughout J.

Hence if $u(x, y)$ satisfies the hypothesis of Theorem A so that ϵ can be taken equal to $K\omega(1/m + 1/n)$, it is certain that P_{mn} will converge uniformly in J to the value of u as m and n both become infinite. Hence we can state

THEOREM III. *In problem B in the case $r > 1$, if $u(x, y)$ satisfies the hypothesis of Theorem A, there will exist a positive constant D_3 independent of m and n, and a positive quantity $\epsilon_{mn} = \omega(1/m + 1/n)$ such that the relation*

$$|u - P_{mn}| \leqq D_3\epsilon_{mn}$$

holds uniformly throughout J, with $\lim_{m,n\to\infty} \epsilon_{mn} = 0$.

6. *Problem B. Convergence in the general case $r > 0$.* Let

$$F(x, y) = u(x, y) - p_{mn}(x, y),$$

where p_{mn} is the polynomial of degrees m, n satisfying relations (10) and (11),

(10) $$|\partial^{i+j}(F)/\partial x^i \partial y^j| \leqq \epsilon, \qquad (i+j=0,1,2),$$

(11) $$|\nabla^2(F)| \leqq 2\epsilon.$$

Then if Q_{mn} is the polynomial of best approximation to F of degrees m, n, and q is the smaller of the two numbers r, s, we can write

$$\begin{aligned}\gamma &= \iint_J |\nabla^2(F-Q_{mn})|^r\,dx\,dy + \lambda \max|F(\alpha,\beta) - Q_{mn}(\alpha,\beta)|^s \\ &\leqq \iint_J |\nabla^2 F|^r\,dx\,dy + \lambda \max|F(\alpha,\beta)|^s \\ &\leqq A(2\epsilon)^r + \lambda\epsilon^s \leqq (A+\lambda)(2\epsilon)^q.\end{aligned}$$

Hence, since each term of γ is $\leqq \gamma$,

(14) $$\iint_J |\nabla^2(F-Q_{mn})|^r\,dx\,dy \leqq (A+\lambda)(2\epsilon)^q,$$

(15) $$\max|F(\alpha,\beta) - Q_{mn}(\alpha,\beta)| \leqq [(A+\lambda)(2\epsilon)^q/\lambda]^{1/s}.$$

So also, by reason of (15) and (10),

(16) $$|Q_{mn}(\alpha,\beta)| \leqq [(A+\lambda)(2\epsilon)^q/\lambda]^{1/s} + \epsilon.$$

Let δ again denote the maximum value of $|\nabla^2 Q_{mn}|$ in J, and let (x_0, y_0) be a point of the region at which $|\nabla^2 Q_{mn}(x_0, y_0)| = \delta$. Then we can show, exactly as in section 4, that a constant E independent of m and n can be found such that

(17) $$\delta \leqq E(\bar{m}^4 \epsilon^q)^{1/r}.$$

where $\bar{m}$ is the greater of the two numbers m, n. Moreover Q_{mn} can be written in terms of the Green's function,

$$Q_{mn}(x,y) = \iint_J G(x,y;\xi,\eta)\nabla^2 Q_{mn}(\xi,\eta)\,d\xi\,d\eta + \chi(x,y),$$

so that throughout J

$$|Q_{mn}(x,y)| \leqq \delta \iint_J |G|\,d\xi\,d\eta + \max|\chi(x,y)|.$$

But $\chi(x,y)$ is harmonic in J and therefore acquires its maximum values on the boundary C. Moreover $\chi(\alpha,\beta) = Q_{mn}(\alpha,\beta)$, so that $\max|\chi(x,y)| = \max|Q_{mn}(\alpha,\beta)|$, and therefore, by (16),

$$\max|\chi(x,y)| \leqq [(A+\lambda)(2\epsilon)^q/\lambda]^{1/s} + \epsilon \leqq M'\epsilon^{q/s},$$

where M' is a constant independent of m and n. By reason of this and (17) it follows that

$$| Q_{mn} | \leqq \left[\iint_J | G | \, d\xi \, d\eta \right] E(\bar{m}^4 \epsilon^q)^{1/r} + M' \epsilon^{q/s}.$$

Hence if P_{mn} is the polynomial of best approximation to u of degrees m, n, we can write

$$\begin{aligned} | u - P_{mn} | &= | F - Q_{mn} | \leqq | F | + | Q_{mn} | \\ &\leqq \epsilon + E(\bar{m}^{4/r} \epsilon^{q/r}) \left[\iint_J | G | \, d\xi \, d\eta \right] + M' \epsilon^{q/r} \leqq E' \bar{m}^{4/r} \epsilon, \end{aligned}$$

from which it follows that if u satisfies the hypothesis of Theorem B with the integer k taken $\geqq 4/r$, the process converges. Thus we have established

THEOREM IV. *In problem B in the general case $r > 0$, if the curve C is subject to the limitations imposed by the requirements of Markoff's theorem, and if $u(x, y)$ satisfies the hypothesis of Theorem B with k taken $\geqq 4/r$, then there will exist a positive quantity $\epsilon_{mn} = \bar{m}^{4/r}(1/m + 1/n)^k \, \Omega(1/m + 1/n)$, and a positive constant D_4 independent of m and n, such that the relation*

$$| u - P_{mn} | \leqq D_4 \epsilon_{mn}$$

holds uniformly throughout J, and furthermore, provided m and n maintain the same order of magnitude,

$$\lim_{m, n \to \infty} \epsilon_{mn} = 0.$$

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ON THE INVERSION FORMULA FOR FOURIER-STIELTJES TRANSFORMS IN MORE THAN ONE DIMENSION. II.

By E. K. Haviland.

A proof of the Continuity Theorem for multi-dimensional Fourier-Stieltjes transforms based on previous results of the author will be given in the present note. This proof,† which for simplicity is given in the case of two dimensions, is believed to be substantially clearer and more direct than the proofs previously given,‡ the improvement being made possible on the one hand by the use of the Convolution Theorem for Fourier-Stieltjes transforms, first proved generally by the author,§ and on the other hand by the use of the inversion formula recently proved by the author.¶ A previous proof ‖ of particular results contained in the complete Convolution Theorem was based on the Continuity Theorem, while the present author's proof of the complete Convolution Theorem is quite independent of it.

We begin by proving a

Uniqueness Lemma.†† *Let* (i) $f(x, y)$ *be continuous in* $(-\infty < x < +\infty;\ -\infty < y < +\infty)$,

(ii) $\int\int_S |f(x,y)|\, dxdy < +\infty$, *where* S *denotes the entire* (xy)*-plane,*

(iii) $\int\int_S \exp\{i(sx + ty)\} f(x,y)\, dxdy = 0$ *for every real* (s, t).

Then $f(x, y) \equiv 0$.

† The present proof has been developed from a proof of the Continuity Theorem in the one-dimensional case given by A. Wintner in a class on the theory of probability.

‡ For the one-dimensional case, cf. P. Lévy, *op. cit.*; for the multi-dimensional case, cf. V. Romanovsky, *loc. cit.*, p. 41, and S. Bochner, *loc. cit.*, p. 403. The references are collected at the end of the paper.

§ Cf. E. K. Haviland, *loc. cit.* II, p. 651, Theorem V.

¶ Cf. E. K. Haviland, *loc. cit.* III.

Professor C. R. Adams has kindly called my attention to the fact that a statement by B. H. Camp, to the effect that a bounded monotone function is not necessarily of bounded variation, was not intended to refer to functions satisfying *all* the conditions (14) of Hardy to which Camp refers, but that Camp's statement, in its intended sense, is correct, contrary to a remark of the present author in a footnote on p. 95 of the foregoing paper.

‖ Cf. S. Bochner, *ibid.*; cf. in this connection E. K. Haviland, *loc. cit.* II, p. 626.

†† The method of this proof is largely an adaptation of the treatment of a similar problem in one dimension by G. Pólya, *loc. cit.*, pp. 105-106. Cf. also E. K. Haviland, *loc. cit.* II, pp. 638-641.

Proof. Let there be given a rectangle R_1 which may, without loss of generality, be taken to be $(0 \leqq x < \xi;\ 0 \leqq y < \eta)$. A function $g_\delta(x, y)$ is defined as follows: $g_\delta(x, y) = 0$ at those points of the rectangle R_2: $(0 \leqq x \leqq U;\ 0 \leqq y \leqq V)$, where $U > \xi$, $V > \eta$, which are not in R_1; also, $g_\delta(x, y) = 1$ in R_3: $(\delta \leqq x \leqq \xi - \delta;\ \delta \leqq y \leqq \eta - \delta)$, where $0 < \delta < \mathrm{Min}(\xi/2, \eta/2)$; finally, the value of $g_\delta(x, y)$ at a point (x, y) of $R_1 - R_3$ is given by that point of a truncated pyramid having R_1 as base and R_3 as top whose projection is (x, y). This function $g_\delta(x, y)$ is extended to the whole plane by prescribing for it the periods U in x and V in y.

As $g_\delta(x, y)$ is continuous everywhere in S, by the two-dimensional Weierstrass trigonometric approximation theorem † there exists a trigonometric polynomial,

$$P_\epsilon(x, y) = \sum_{-M}^{M} \sum_{-N}^{N} \alpha_{mn} \exp\{i(2\pi mx/U + 2\pi ny/V)\},$$

such that

$$(1) \qquad | g_\delta(x, y) - P_\epsilon(x, y) | < \epsilon$$

for all (x, y). Setting $s = 2\pi m/U$, $t = 2\pi m/V$ in (iii), we see that

$$\iint_S \exp\{i(2\pi mx/U + 2\pi ny/V)\} f(x, y)\, dxdy = 0.$$

Hence $\iint_S P_\epsilon(x, y) f(x, y)\, dxdy = 0$. We first let $\epsilon \to 0$ in (1). Since

$$\iint_{S-R} | P_\epsilon(x, y) f(x, y) |\, dxdy \leqq 2 \iint_{S-R} | f(x, y) |\, dxdy < \eta,$$

where η is arbitrarily small, provided $\epsilon\ (> 0)$ is sufficiently small and the rectangle R sufficiently large, it follows from the Arzelà-Lebesgue theorem that

$$\iint_S P_\epsilon(x, y) f(x, y)\, dxdy \to \iint_S g_\delta(x, y) f(x, y)\, dxdy, \qquad (\epsilon \to 0),$$

so that the latter integral vanishes.

In the second place, we let $\delta \to 0$, whereupon $g_\delta(x, y) \to g(x, y)$, a function equal to one within R_1 and its periodic images and to zero elsewhere. Let the rectangle R_1 now be denoted by R_{10} and let R_{1i}, $(i = 1, 2, 3, \cdots)$, be periodic images of R_{10}. If R_{2i} be the periodic image of R_2 containing R_{1i}, it follows from (ii) and the inequality $| g_\delta(x, y) | \leqq 1$ that

$$(2) \qquad \sum_{i=0}^{\infty} \iint_{R_{2i}} g_\delta(x, y) f(x, y)\, dxdy = \iint_S g_\delta(x, y) f(x, y)\, dxdy = 0.$$

By again applying the Arzelà-Lebesgue Theorem, we find for every fixed ν

† Cf. L. Tonelli, *op. cit.*, p. 494.

$$(3) \quad \sum_{i=0}^{\nu} \int\int_{R_{2i}} g_\delta(x,y) f(x,y)\,dxdy \to \sum_{i=0}^{\nu} \int\int_{R_{1i}} f(x,y)\,dxdy, \qquad (\delta \to 0),$$

by the definition of $g(x,y)$. Since this implies that the integral on the right of (3) is zero, on letting $\nu \to \infty$, it follows that

$$\sum_{i=0}^{\infty} \int\int_{R_{1i}} f(x,y)\,dxdy = 0.$$

Finally, we let $U \to +\infty$, $V \to +\infty$, whereupon we obtain, in view of the absolute convergence of the foregoing series and of the continuity of $f(x,y)$ in R_{10},

$$\int_0^\xi dx \int_0^\eta f(x,y)\,dy = \int\int_{R_{20}} f(x,y)\,dxdy = 0.$$

Since we may choose (ξ,η) arbitrarily and since $f(x,y)$ is continuous, we may differentiate the latter integral with respect to ξ and η, obtaining $f(\xi,\eta) \equiv 0$, q. e. d. It is to be noted that the hypothesis (i) was used only in the final step of the proof of the lemma.

We are now in a position to prove the

CONTINUITY THEOREM FOR FOURIER-STIELTJES TRANSFORMS.† *If* $\{\phi_n\}$ *be a sequence of distribution functions and* $\{\Lambda(s,t;\phi_n)\}$ *the sequence of corresponding Fourier-Stieltjes transforms, then a necessary and sufficient condition that the sequence* $\{\phi_n\}$ *should converge to a distribution function* ϕ *is that the sequence* $\{\Lambda(s,t;\phi_n)\}$ *converges to a function* $h(s,t)$ *uniformly in every finite region of the* (s,t)*-plane. Furthermore,* $h(s,t) = \Lambda(s,t;\phi)$.

Proof. We first prove the sufficiency of the condition, noting that as a consequence of our hypothesis $h(s,t)$ is continuous at every point of the (s,t)-plane and $|h(s,t)| \leqq 1$.

Let $\gamma(E)$ be the two-dimensional Gaussian distribution function; i. e., to $\gamma(E)$ corresponds ‡ the point function

$$G(x,y) = (2\pi)^{-1} \int_{-\infty}^{x} \int_{-\infty}^{y} \exp\{-(\xi^2+\eta^2)/2\}\,d\xi d\eta.$$

As the Fourier-Stieltjes transform of $\gamma(E)$ may be regarded as an iterated integral, its value may be computed from the known result in the case of one dimension to be

$$(4) \qquad \Lambda(s,t;\gamma) = \exp\{-(s^2+t^2)/2\}.$$

We then set

† For definition of terms occurring in this theorem, cf., e. g., E. K. Haviland, *loc cit.* II, pp. 627-628.

‡ Cf. J. Radon, *loc. cit.*, p. 1304; E. K. Haviland, *loc. cit.* II, p. 627.

(5) $$L(s,t)=h(s,t)\Lambda(s,t;\gamma).$$

It is a continuous function of (s,t) and

(6) $$|L(s,t)|\leqq|\Lambda(s,t;\gamma)|=\Lambda(s,t;\gamma).$$

Let $\{\phi_{m_n}\}$ be a convergent subsequence of $\{\phi_n\}$ and $\tau=\tau(E)$ be its limit. $\tau(E)$ is monotone by the Compactness Theorem of Radon † and $0\leqq\tau(E)\leqq 1$ for all E. We next put ‡

(7) $$\rho_n=\phi_{m_n}*\gamma$$

and

(7ᵃ) $$\rho=\tau*\gamma.$$

Since §

$$\Lambda(s,t;\rho_n)=\Lambda(s,t;\phi_{m_n})\cdot\Lambda(s,t;\gamma) \text{ and } |\Lambda(s,t;\phi_{m_n})|\leqq\int\int_S d_{xy}\phi_{m_n}(E)=1,$$

it follows that

(8) $$|\Lambda(s,t;\rho_n)|\leqq\Lambda(s,t;\gamma),$$

uniformly with respect to n. Similarly,

$$|\Lambda(s,t;\rho)|\leqq\Lambda(s,t;\gamma).$$

ρ_n and ρ are both continuous by virtue of the addition rule of line spectra. We proceed to show that, as $n\to\infty$, $\rho_n(R)\to\rho(R)$ for *every* rectangle R. Not only does

$$\rho_n(R)=\int\int_S\phi_{m_n}(R-P_{xy})\exp\{-(x^2+y^2)/2\}dxdy$$

exist for every R, due to the continuity of γ, but the integrand of ρ_n has a bounded and absolutely integrable majorant independent of n. Also, $\phi_{m_n}(E)\to\tau(E)$ on all non-singular lines of the latter as $n\to\infty$. Then, by the Arzelà-Lebesgue Theorem, as $n\to\infty$,

(10) $$\rho_n(R)\to\int\int_S\tau(R-P_{xy})\exp\{-(x^2+y^2)/2\}dxdy=\tau*\gamma=\rho.$$

Since ρ_n and ρ have no singular rectangles, we obtain by the inversion formula for Fourier-Stieltjes transforms ¶

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† Cf. E. K. Haviland, *loc. cit.* I, p. 551.

‡ $\psi_1*\psi_2$ denotes the symbolical product (Faltung or convolution) of ψ_1 and ψ_2. It is sufficient for its existence that ψ_1 and ψ_2 be monotone bounded functions, in which case the addition rule of spectra also holds. Cf. E. K. Haviland, *loc. cit.* II, p. 654.

§ This follows from the Convolution Theorem for Fourier-Stieltjes transforms. Cf. E. K. Haviland, *loc. cit.* II, p. 651, Theorem V. It is important for what follows to note that the theorem holds for any two arbitrary monotone bounded functions.

¶ Cf. E. K. Haviland, *loc. cit.* III, p. 99, equation (8).

$$(11)\quad -(2\pi)^2\rho_n(R) = \int\int_S (st)^{-1}\Lambda(s,t;\rho_n)[e^{-is\xi}-1][e^{-it\eta}-1]e^{-i(us+vt)}dsdt,$$

$$(12)\quad -(2\pi)^2\rho(R) = \int\int_S (st)^{-1}\Lambda(s,t;\rho)[e^{-is\xi}-1][e^{-it\eta}-1]e^{-i(us+vt)}dsdt.$$

It is not necessary to use Cauchy principal values in these equations, as both $\Lambda(s,t;\rho_n)$ and $\Lambda(s,t;\rho)$ possess absolutely integrable majorants in virtue of (8), (9) and (4). It follows from (10) that, as $n\to\infty$, the left-hand side of (11) approaches the left-hand side of (12). In consequence, the right-hand side of (11) must approach the right-hand side of (12).

Now from (4), (6) and (8), together with the fact that $[e^{-is\xi}-1]/s$, $[e^{-it\eta}-1]/t$ are uniformly bounded for all (s,t), it follows that the Arzelà-Lebesgue Theorem may be applied to the right-hand side of (11), so that, as $n\to\infty$,

$$(13)\quad -(2\pi)^2\rho_n(R)\to\int\int_S (st)^{-1}L(s,t)[e^{-is\xi}-1][e^{-it\eta}-1]e^{-i(us+vt)}dsdt$$

in virtue of (5), (7) and the Convolution Theorem for Fourier-Stieltjes transforms. Hence, by the last remark of the preceding paragraph,

$$(14)\quad \int\int_S f(s,t)(st)^{-1}[e^{-is\xi}-1][e^{-it\eta}-1]e^{-i(us+vt)}dsdt=0,$$

where $f(s,t)=L(s,t)-\Lambda(s,t;\rho)$, so $|f(s,t)|\leqq 2\Lambda(s,t;\gamma)$, which implies the absolute integrability of the integrand in (14). Then we may differentiate with respect to ξ and η beneath the integral sign in (14), obtaining

$$(15)\quad \int\int_S f(s,t)\exp\{-i[s(\xi+u)+t(\eta+v)]\}dsdt=0.$$

From (5) and from the definition of $f(s,t)$, together with the fact that (15) holds for all $(\xi+u)$, $(\eta+v)$, it follows that $f(s,t)$ satisfies the conditions of the Uniqueness Lemma, so $f(s,t)\equiv 0$, or $L(s,t)\equiv\Lambda(s,t;\rho)$, or by (5) and (7), as $\Lambda(s,t;\gamma)\neq 0$,

$$(16)\quad h(s,t)=\Lambda(s,t;\tau).$$

Consequently, $\Lambda(s,t;\tau)$ does not depend on the special choice of the subsequence $\{\phi_{m_n}\}$ and as (by the inversion formula) τ is determined up to its singular lines by its Fourier-Stieltjes transform, it follows that τ does not depend on the special choice of $\{\phi_{m_n}\}$. This implies that $\{\phi_n\}$ is convergent, for otherwise it would be possible to select from $\{\phi_n\}$ two subsequences converging to essentially distinct limits, say τ_1 and τ_2. As, however, τ_1 and τ_2

have the same Fourier-Stieltjes transforms, this leads to a contradiction. Finally, if we set $s = t = 0$ in (16), we see that $\Lambda(0, 0; \tau) = 1$, so τ is indeed a distribution function. As τ may thus be taken as ϕ, this completes the proof of the first half of the theorem.

To prove the second half of the theorem, we set $\exp\{i(sx + ty)\} = g(s, t; x, y)$ and let J be a non-singular square of ϕ so large that

$$(17) \qquad 0 \leqq \phi(S - J) < \epsilon.$$

Then let N'_ϵ be chosen so large that $|\phi_n(S - J) - \phi(S - J)| < \epsilon$ for all $n \geqq N'_\epsilon$. It follows that for all such n

$$(18) \qquad 0 \leqq \phi_n(S - J) < 2\epsilon.$$

We next take a division of J: $(-M \leqq x \leqq M; -M \leqq y \leqq M)$ by drawing parallels to the axes, these parallels being non-singular lines of ϕ and dividing J into a finite number, m, of rectangles R_k whose greatest diameter is δ_m, $\lim_{m=\infty} \delta_m = 0$. By choosing $\delta_m < \delta = \delta(\epsilon)$, we can make

$$|g(s, t; x_k, y_k) - g(s, t; x'_k, y'_k)| < \epsilon,$$

where (x_k, y_k), (x'_k, y'_k) are any two points of R_k, and $\delta(\epsilon)$ is independent of (s, t) in an arbitrarily fixed closed rectangle Σ of the st-plane. Then if $\delta_m < \delta$, we have †

$$(19) \qquad |\sum_{k=1}^{m} g(s, t; x_k, y_k)\phi_n(R_k) - \int\int_J g(s, t; x, y)\, d_{xy}\phi_n(E)| < \epsilon \int\int_S d_{xy}\phi_n(E) = \epsilon,$$

and similarly

$$(20) \qquad |\sum_{k=1}^{m} g(s, t; x_k, y_k)\, \phi(R_k) - \int\int_J g(s, t; x, y)\, d_{xy}\phi(E)| < \epsilon.$$

But m being fixed when δ is chosen and the m rectangles R_k being non-singular rectangles of ϕ,

$$(21) \qquad |\sum_{k=1}^{m} g(s, t; x_k, y_k)\phi(R_k) - \sum_{k=1}^{m} g(s, t; x_k, y_k)\phi_n(R_k)| \leqq \sum_{k=1}^{m} |\phi(R_k) - \phi_n(R_k)| < \epsilon,$$

provided $n \geqq N''_\epsilon$. Hence if $N_\epsilon = \operatorname{Max}(N'_\epsilon, N''_\epsilon)$, it follows from (17), (18), (19), (20), (21) that

$$|\int\int_S g(s, t; x, y)\, d_{xy}\phi(E) - \int\int_S g(s, t; x, y)\, d_{xy}\phi_n(E)| = |\Lambda(s, t; \phi) - \Lambda(s, t; \phi_n)| < 6\epsilon,$$

† Cf. J. Radon, *loc. cit.*, p. 1324, equation (14).

provided $n \geqq N_\epsilon$, where N_ϵ is independent of (s, t) in the arbitrarily fixed rectangle Σ, q. e. d.

COROLLARY. *If, as $n \to \infty$, the sequence of Fourier-Stieltjes transforms $\{\Lambda(s, t; \phi_n)\}$ converges in the whole (s, t)-plane to a continuous function $h(s, t)$ then the convergence is uniform in every finite region of the (s, t)-plane.*

Proof. Bochner has shown † that the convergence of $\{\Lambda(s, t; \phi_n)\}$ to a continuous function $h(s, t)$ is a sufficient condition for the convergence of $\{\phi_n\}$ to a distribution function ϕ, while we have shown that the uniform convergence of the sequence of Fourier-Stieltjes Transforms in every finite region of the (s, t)-plane is both a necessary and a sufficient condition for the essential convergence of $\{\phi_n\}$ to a distribution function ϕ. Thus Bochner's statement of the Continuity Theorem ‡ is in reality no more general than the usual § formulation of the theorem.

REFERENCES.

S. Bochner, "Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse," *Mathematische Annalen,* vol. 108 (1933), pp. 378-410.

E. K. Haviland, I: "On statistical methods in the theory of almost-periodic functions," *Proceedings of the National Academy of Sciences,* vol. 19 (1933), pp. 549-555.

II: "On the theory of absolutely additive distribution functions," *American Journal of Mathematics,* vol. 56 (1934), pp. 625-658.

III: "On the inversion formula for Fourier-Stieltjes transforms in more than one dimension," *ibid.,* vol. 57 (1935), pp. 94-100.

P. Lévy, *Calcul des probabilités* (Paris, 1925).

G. Pólya, "Herleitung des Gausssschen Fehlergesetzes aus einer Funktionalgleichung," *Mathematische Zeitschrift,* vol. 18 (1923), pp. 96-108.

J. Radon, "Theorie und Anwendungen der absolut additiven Mengenfunktionen," *Sitzungsberichte der mathematischen-naturwissenschaftlichen Klasse der Kaiserl. Akademie zu Wien,* vol. 122 (1913), pp. 1295-1438.

V. Romanovsky, "Sur un théorème limite du calcul des probabilités," *Recueil Mathématique de la Société Mathématique de Moscou,* vol. 36 (1929), pp. 36-64.

L. Tonelli, *Serie Trigonometriche* (Bologna, 1928).

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† Cf. S. Bochner, *loc. cit.,* p. 403, Theorem 17. In this connection, it may be noted that our proof for our sufficient condition may be used without modification to prove Bochner's Theorems 16 and 17, save that in the former case the integrals must be considered as Lebesgue integrals, so that from the differentiation of our equation (14) we may conclude only that $f(s, t) = 0$ almost everywhere.

‡ Cf. J. Radon, *loc. cit.,* p. 1324, equation (14).

§ Cf. P. Lévy, *op. cit.*

ISOLATED CRITICAL POINTS.

By Arthur B. Brown.

The object of this note is to replace an incomplete proof of an earlier paper * by a proof using the methods of that paper. Professor Marston Morse, originator of the general theory of critical points, who pointed out to the writer that in the proof of Lemma 14 of BI it is not shown that a deformation is determined, has published results of which this Lemma 14 is a corollary.† The treatment‡ to follow is of different nature from the treatment of the point in question by Morse.

Proof of Lemma 14. We subdivide the complex D (defined on page 265 of BI) regularly at least once till the D-neighborhoods,§ say $\mathcal{N}_d$, of the centers P of the spheres S, with boundaries, are interior to the spheres S. If we remove the points P from $\mathcal{N}_d$, then the remainder, $\mathcal{N}'_d$, of $\mathcal{N}_d$ is covered by a field $\mathcal{F}$ of curves, each curve joining a point P to a point of $W = \bar{\mathcal{N}}_d - \mathcal{N}_d$, as follows easily from the structure of a simplicial complex. Let B'' be the set defined like B', but for smaller spheres, say S_2, so that any point of W is outside all the spheres S_2. If we shrink $\mathcal{N}'_d$ down onto W by use of the field $\mathcal{F}$, then the resulting deformation, say (D_1), carries D' over itself into a subset of B''. Points outside $\bar{\mathcal{N}}_d$ remain fixed under (D_1).

Let $[\Sigma]$ be a set of spheres slightly larger than S, concentric with the latter and satisfying the same conditions. Choose $\epsilon > 0$ so small that the

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* A. B. Brown, "Relations between the critical points of a real analytic function of n independent variables," *American Journal of Mathematics*, vol. 52 (1930), pp. 251-270. We refer to this paper as BI. Cf. footnote 3 of the writer's paper, "Critical sets of an arbitrary real analytic function of n variables," *Annals of Mathematics*, vol. 32 (1931), pp. 512-520.

† Marston Morse, "The critical points of a function of n variables," *Transactions of the American Mathematical Society*, vol. 33 (1931), pp. 72-91 (Morse I). Lemma 14 of BI is a corollary of Theorem 9, page 84, of Morse I. See also Theorem 5. I, page 156, of Marston Morse, "The calculus of variations in the large," *American Mathematical Society Colloquium Publications*, vol. 18, New York, 1934 (Morse II). For other papers on critical points see bibliography of Morse II.

‡ The writer does not know whether the questionable statement in the "proof" of Lemma 14 in BI is or is not true. Shortly before the appearance of Morse's *Colloquium* the writer, having momentarily forgotten that Lemma 14 follows from results in Morse I, devised the present proof.

§ That is, sets of all cells of D having a vertex at any of the points P. For notations in topology see S. Lefschetz, "Topology," *American Mathematical Society Colloquium Publications*, vol. 12, New York, 1930. That complexes in the sense of analysis situs are at hand is proved by B. O. Koopman and A. B. Brown, *Transactions of the American Mathematical Society*, vol. 34 (1932), pp. 231-251; also by S. Lefschetz and J. H. C. Whitehead, *ibid.*, vol. 35 (1933), pp. 510-517. The fact that complexes are present was used in BI.

trajectories τ (§ 9 of BI)* exist between and on the pairs of spheres Σ and S_2, at points where $c-\epsilon \leqq f \leqq c$. Recall that the parameter for the trajectories τ is the distance r from P, in any sphere Σ. Let a_{s_2} and a_σ denote the radii of S_2 and Σ respectively, and M the minimum distance from the locus $f \leqq c-\epsilon/2$ to the locus $f=c$, between or on the pairs of spheres S_2 and Σ. Consider the transformation which acts only upon the points Q of S_2 satisfying $c-\epsilon \leqq f \leqq c$ sending each such point into a point Q' on the same trajectory τ, and determined by

$$(1) \qquad r' = r + \frac{(f-c+\epsilon)}{\epsilon} \cdot (a_\sigma - a_{s_2}).$$

We now determine a deformation (D_2) which keeps fixed all points except those on the trajectories τ between the pairs of points Q and Q'. The deformation causes each of the trajectories QQ' to shrink down to the point Q', and is defined in an obvious way in terms of r.

Since $f =$ constant on any trajectory τ, it follows from (1) that B'' is carried by (D_2) into a set whose points within distance $\frac{1}{2}(a_\sigma - a_{s_2})$ from S_2 satisfy $f \leqq c-\epsilon/2$, and hence are distant at least M from the part of the locus $f=c$ between or on the spheres S_2 and Σ. Hence we can follow (D_2) by a deformation along radial lines through P in each sphere Σ, affecting only points within distance $U_{s_2} = \min. [M, \frac{1}{2}(a_\sigma - a_{s_2})]$ of S_2, so that, as a result of the two deformations, locus B'' is deformed over itself into a subset of the corresponding locus for spheres of radius $a_{s_2} + U_{s_2}$. It is then clear that a finite number of such steps will deform B'' over itself into a subset of B', with B' remaining on B' during the entire resulting deformation (D_3).

If now we perform (D_1) and then (D_3), it is seen that the resulting deformation (D_4) carries D' into a subset of B', while keeping B' on B'. From Theorem 2, page 252, of BI, it follows that B' and D' have the same Betti numbers, and Lemma 14 is proved.†

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* In the more general case treated by Morse, the trajectories τ become the (ϕf)-trajectories (Morse I, page 80; Morse II, page 153). The $(f\phi)$-trajectories of Morse's treatment do not appear in BI.

† We wish also to point out that on page 261 of BI, the definition of configuration is not given properly. In lines 7 to 2 from the bottom, "when $n-s$ . . . . (ordinary points)" should be replaced by "when, after a non-singular linear transformation, $n-s$ of the variables are the values of the dependent variables defined by the vanishing of $n-s$ algebroid functions (pseudopolynomials), where the other s variables, say $\xi_1, \cdots, \xi_s$, are the independent variables for each algebroid function. These values are analytic at points where the discriminants of the algebroid functions are not zero (ordinary points)". On page 262, line 4, "$x_1, \cdots, x_s$. Therefore if the" is replaced by "$\xi_1, \cdots, \xi_s$. Therefore if a". In line 6, "discriminant" is replaced by "discriminants, separately and severally,". In line 12, delete "variables $x_1, x_2, \cdots, x_n$ as".

CYCLOTOMY, HIGHER CONGRUENCES, AND WARING'S PROBLEM.

By L. E. Dickson.

1. *Introduction.* This memoir does not presuppose any knowledge of the subjects treated. The outstanding Waring problem is to find s such that every large integer is a sum of s positive integral values of a given polynomial. An account of its recent solution is given in § 29. One step is the proof that every integer is congruent to a sum of s values of the polynomial with respect to every prime modulus p and certain powers of p. The proof employs the number N of solutions of $x^e + y^e + 1 \equiv 0 \pmod{p = ef + 1}$.

By far the most effective method of finding * N is that of cyclotomy, which yields also the number of solutions of any trinomial congruence involving three e-th powers multiplied by any integers.

The periods can be expressed by radicals in terms of certain resolvent functions. But this algebraic side of cyclotomy has little practical application to our problem to find the e^2 cyclotomic constants (k, h), which are coefficients in the product of two periods expressed linearly in terms of the periods.

Unfortunately the latter problem has been solved heretofore only when $e \leqq 5$ and then the problem is so simple † that there arise none of the difficulties for $e \geqq 6$.

While $e = 6$ had been treated, the solution involved the six numbers $A, \cdots, F$ in the decompositions

$$p = A^2 + 3B^2, \qquad 4p = L^2 + 27M^2, \qquad 4p = E^2 + 3F^2,$$

whereas the true solution (§ 17) of the problem involves only A and B. A similarly perfect solution is obtained for the new cases $e = 8$, 10, 12. The odd values of e are not needed in Part 2.

Our methods serve for further values of e. But the results must be postponed to later papers.

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* We need a formula for N which implies that N will exceed any given number when p exceeds an obtainable limit. In *Journal de Mathematiques*, vol. 2 (1837), pp. 253-292, V. A. Lebesgue found that N is congruent modulo p to a long sum of binomial coefficients. But this result does not yield the needed property.

† Except for the proof when $e = 5$ that the pair of Diophantine equations have essentially a unique solution.

PART I. CYCLOTOMY, HIGHER CONGRUENCES.

2. *The periods.* Let g be a primitive root of a prime p. Let e be a divisor of $p-1$ and write $p-1=ef$. Let R be any (imaginary) root $\neq 1$ of $x^p=1$. The sums

$$\eta_k=\sum_{t=0}^{f-1} R^{g^{et+k}} \qquad (k=0,1,\cdots,e-1) \tag{1}$$

are called *periods*. For example, if $p=7$, $e=3$, then $f=2$ and 3 is a value of g. Since $g^2\equiv 2$, $g^3\equiv 6 \pmod 7$, the periods (1) are

$$\eta_0=R+R^6, \qquad \eta_1=R^3+R^4, \qquad \eta_2=R^2+R^5.$$

Let s be the summation index for η_0. For a fixed s, we may replace t in (1) by $t+s$, which ranges with t over a complete set of residues modulo f. Hence

$$\eta_0\eta_k=\sum_{s=0}^{f-1}\sum_{t=0}^{f-1} R^{g^{es}N}, \qquad (N=1+g^{et+k}). \tag{2}$$

First, let $N\equiv 0 \pmod p$. Since $0\leqq et+k\leqq ef-1\leqq p-2$,

$$et+k=\tfrac{1}{2}(p-1).$$

If f is even, k is divisible by e, whence $k=0$, $t=f/2$. But if f is odd, k is divisible by $e/2$, while $k\neq 0$ since $ef/2$ is not divisible by e, whence $k=e/2$, $t=(f-1)/2$. Make the definition

(3) $n_k=1$ if f is even and $k=0$, or if f is odd and $k=e/2$;
$n_k=0$ in all remaining cases.

Hence $N\equiv 0 \pmod p$ holds for exactly n_k values of t, and the corresponding part of (2) is fn_k.

Second, let N be prime to p, whence N is congruent to a power of the primitive root g:

$$1+g^{et+k}\equiv g^{ez+h} \pmod p, \tag{4}$$

where $0\leqq h\leqq e-1$, $0\leqq z\leqq f-1$. When h (as well as k) is fixed, let

(5) (k,h) be the number of sets of values of t and z,
each chosen from $0,1,\cdots,f-1$, for which (4) holds.

Hence (k,h) is unaltered if we increase (or decrease) either k or h by any multiple of e. For fixed values of t and z satisfying (4), the corresponding part of (2) is

$$\sum_{s=0}^{f-1} R^{g^{e(s+z)+h}} = \sum_{s=0}^{f-1} R^{g^{es+h}} = \eta_h,$$

since $s+z$ ranges with s over a complete set of residues modulo f. This completes the proof of

$$\eta_0\eta_k = \sum_{h=0}^{e-1} (k,h)\eta_h + fn_k \qquad (k=0,1,\cdots,e-1). \tag{6}$$

Replace R by R^{g^m}. Then η_k becomes η_{k+m}, in which we may reduce subscripts of η modulo e. Hence

$$\eta_m\eta_{m+k} = \sum_{h=0}^{e-1} (k,h)\eta_{m+h} + fn_k. \tag{7}$$

3. *The period equation.* Since the e periods (1) contain without duplication $R,\cdots,R^{p-1}$, whose sum is -1,

$$1+\eta_0+\eta_1+\cdots+\eta_{e-1}=0. \tag{8}$$

Employ also (6) for $k=1,\cdots,e-1$. Regard η_0 as a constant. We have e linear homogeneous equations in $1,\eta_1,\cdots,\eta_{e-1}$. Hence

$$\begin{vmatrix} 1+\eta_0 & 1 & \cdots & 1 \\ fn_1+(1,0)\eta_0 & (1,1)-\eta_0 & \cdots & (1,e-1) \\ \cdot & \cdot & \cdot & \cdot \\ fn_{e-1}+(e-1,0)\eta_0 & (e-1,1) & \cdots & (e-1.e-1)-\eta_0 \end{vmatrix} = 0, \tag{9}$$

which is the period equation satisfied by η_0 and also by every η_k.

4. *Auxiliary congruence.* The number of sets of values of t and z, each chosen from $0,1,\cdots,f-1$, which satisfy

$$1+g^{et+k}+g^{ez+h}\equiv 0 \pmod{p} \tag{10}$$

will be denoted by $\{k,h\}=\{h,k\}$. Evidently $\{k,h\}$ is unaltered if we increase k and h by multiples of e. Multiply (10) by the reciprocal of its second term; we get

$$1+g^{e(-t)-k}+g^{e(z-t)+h-k}\equiv 0 \pmod{p}.$$

Since $-t$ and $z-t$ uniquely determine t and z modulo f,

$$\{-k,h-k\}=\{k,h\}. \tag{11}$$

We may express $\{k,h\}$ in terms of our former (i,j). First, let f be even. Then

$$p-1=2e\cdot f/2, \qquad -1\equiv g^{(p-1)/2}=g^{e\cdot f/2} \pmod{p}.$$

Thus (10) may be written as

$$1 + g^{et+k} \equiv g^{e(z+f/2)+h} \pmod p.$$

Comparison with (4) gives

$$\{k, h\} = (k, h), \qquad f \text{ even.} \tag{12}$$

For f odd, (10) may be written as

$$1 + g^{et+k} \equiv g^m \pmod p, \qquad m = e[z + \tfrac{1}{2}(f-1)] + h + \tfrac{1}{2}e,$$

$$\{k, h\} = (k, h + \tfrac{1}{2}e), \qquad f \text{ odd.} \tag{13}$$

From $\{k, h\} = \{h, k\}$, (11), (12), (13), we get

$$(k, h) = (h, k), \qquad (e-k, h-k) = (k, h), \qquad f \text{ even,} \tag{14}$$

$$(k, h) = (h + \tfrac{1}{2}e, k + \tfrac{1}{2}e), \qquad (e-k, h-k) = (k, h), \qquad f \text{ odd.} \tag{15}$$

By (12) and (13), the systems (14) and (15) are permuted when

$$(k, h) \text{ corresponds to } (k, h + \tfrac{1}{2}e). \tag{16}$$

5. *Linear relations.* The sum (2) involves f^2 powers of R. In (6) the number of powers of R (including 1) is $\Sigma(k, h)f + fn_k$. Cancelling f, we get

$$\sum_{h=0}^{e-1} (k, h) = f - n_k \qquad (k = 0, 1, \cdots, e-1). \tag{17}$$

It may be verified by (14) and (15) that we may discard as redundant those relations (17) in which $k > e/2$ if e is even, but $k > (e-1)/2$ if e is odd and hence f even.

6. *Case* $e = 2$. We employ (3), (14)-(17). For f even,

$$\begin{aligned} &(0, 0) + (0, 1) = f - 1, \qquad (1, 0) + (1, 1) = f, \\ &(1, 1) = (1, 0) = (0, 1) = f/2, \qquad (0, 0) = \tfrac{1}{2}f - 1. \end{aligned} \tag{18}$$

For f odd,

$$\begin{aligned} &(0, 0) + (0, 1) = f, \qquad (1, 0) + (1, 1) = f - 1, \\ &(0, 0) = (1, 1) = (1, 0) = (f-1)/2, \qquad (0, 1) = (f+1)/2. \end{aligned} \tag{19}$$

Hence for every f, the (ij) are uniquely determined by $p = 2f + 1$. The period equation (9) is $\eta^2 + \eta + c = 0$, where $c = fn_1 - (1, 1)$, $c = -\frac{1}{4}(p-1)$ if f even, $c = \frac{1}{4}(p+1)$ if f odd.

7. When $e \geqq 3$, (14)-(17) do not determine the (k, h), but must be supplemented by relations obtained by the following advanced theory. By (7),

$$\sum_{j=0}^{e-1} \eta_j \eta_{j+k} = \sum_{j=0}^{e-1} \sum_{h=0}^{e-1} (k, h) \eta_{j+h} + efn_k.$$

For a fixed h we may replace j by $j - h$; the double sum becomes

$$\Sigma_j [\Sigma_h (k, h)] \eta_j = \Sigma_j (f - n_k) \eta_j = -(f - n_k),$$

by (17) and (18). Also

$$efn_k - (f - n_k) = (ef + 1) n_k - f = pn_k - f,$$

$$\sum_{j=0}^{e-1} \eta_j \eta_{j+k} = pn_k - f \qquad (k = 0, \cdots, f - 1). \tag{20}$$

8. *Jacobi's functions.* Let α be any root $\neq 1$ of $\alpha^{p-1} = 1$. Write

$$F(\alpha) = \sum_{k=0}^{p-2} \alpha^k R^{g^k}. \tag{21}$$

Usually we employ a special case of this function (21) due to Jacobi. Let $p = ef + 1$ and let β be a primitive e-th root of unity. In (21) take $\alpha = \beta^n$, write $k = et + j$ and employ (1). Thus

$$F(\beta^m) = \sum_{j=0}^{e-1} \beta^{mj} \eta_j. \tag{22}$$

Consider its product by $F(\beta^n)$. For j fixed, the summation index in $F(\beta^n)$ may be taken to be $j + k$, which ranges with k over a complete set of residues modulo e. Hence

$$F(\beta^m) F(\beta^n) = \sum_{k=0}^{e-1} \sum_{j=0}^{e-1} \beta^{mj} \beta^{n(j+k)} \eta_j \eta_{j+k},$$

$$F(\beta^m) F(\beta^n) = \sum_{k=0}^{e-1} \beta^{nk} M_k, \qquad M_k = \sum_{j=0}^{e-1} \beta^{(m+n)j} \eta_j \eta_{j+k}. \tag{23}$$

First, let $m = -n$, where n is not a multiple of e. Thus M_k has the value (20). Since the sum of the n-th powers of the roots $1, \beta, \cdots, \beta^{e-1}$ of $x^e = 1$ is zero,

$$\sum_{j=0}^{e-1} \beta^{nk} = 0. \tag{24}$$

Transpose the term given by $k = 0$ or $k = e/2$ according as f is even or odd, and apply (3). Note that if f is odd, e is even and $\beta^{e/2} = -1$. Hence

$$F(\beta^n) F(\beta^{-n}) = (-1)^{nf} p, \; n \text{ not divisible by } e. \tag{25}$$

Second, let no one of n, m, $n+m$ be divisible by e. Write

$$N_k = \sum_{j=0}^{e-1} \beta^{(m+n)j} \sum_{h=0}^{e-1} (k,h)\eta_{j+h}.$$

By (7) with m replaced by j and (24),

$$M_k - N_k = fn_k \sum_{j=0}^{e-1} \beta^{(m+n)j} = 0.$$

Evidently N_k is the product of

$$F(\beta^{m+n}) = \sum_{j=0}^{e-1} \beta^{(m+n)(j+h)}\eta_{j+h}, \qquad \sum_{h=0}^{e-1} \beta^{-(m+n)h}(k,h).$$

Since the first sum is independent of k,

$$\text{(26)} \qquad \frac{F(\beta^m)F(\beta^n)}{F(\beta^{m+n})} = \sum_{k=0}^{e-1} \beta^{nk} \sum_{h=0}^{e-1} \beta^{-(m+n)h}(k,h) \equiv R(m,n)$$

(no one of m, n, $m+n$ divisible by e).

We may shorten the computation of $R(m,n)$ by combining its terms in pairs. By the second relation in (14) or (15), $(e-j,h) = (j,j+h)$. Hence the part of (26) given by $k=e-j$ with $j \geqq 1$ is equal to

$$\beta^{n(e-j)} \sum_{h=0}^{e-1} \beta^{-(m+n)h}(j,j+h).$$

We may replace the index h by $h-j$ and get

$$\beta^{mj} \sum_{h=0}^{e-1} \beta^{-(m+n)h}(j,h).$$

If $j < e/2$, we may combine this with the new term of R given by $k=j$. Write $E = e/2$ if e is even, $E = (e-1)/2$ if e is odd. Thus

$$\text{(27)} \quad R(m,n) = \sum_{j=1}^{E} (\beta^{mj} + \beta^{nj}) \sum_{h=0}^{e-1} \beta^{-(m+n)h}(j,h) + \sum_{h=0}^{e-1} \beta^{-(m+n)h}(0,h),$$

when e is odd. But for e even, (27) holds only when in the term given by $j=E$ we replace $\beta^{mj}+\beta^{nj}$ by β^{mE} if $m \equiv n \pmod 2$, but by zero if $m \not\equiv n \pmod 2$.

Employ (25) also with n replaced by m and by $m+n$. Then (26) gives

(28) $R(m,n)R(-m,-n) = p$, none of m, n, $m+n$ divisible by e;

(29) $R(-m,-n)$ is derived from $R(m,n)$ by replacing β by β^{-1}.

9. *Case* $e = 3$. For a prime $p = 3f + 1$, f is even. By (14),

$$(30)\quad (10) = (01),\ (11) = (02),\ (20) = (02),\ (21) = (12),\ (22) = (01).$$

Hence the nine (ij) reduce to (00), (01), (02), (12). By (17) and (27),

$$(31)\qquad (00) + (01) + (02) = f - 1,\quad (01) + (02) + (12) = f,$$

$$(32)\quad R(1,1) = u + 3\beta M,\ u = (00) + 2(12) - 3(02),\ M = (01) - (02).$$

By (28) and (29),

$$4p = 4(u + 3\beta M)(u + 3\beta^2 M) = (2u - 3M)^2 + 27M^2.$$

Multiply equations (31) by 2 and 5, and subtract. Thus

$$2(00) - 3(01) - 3(02) - 5(12) = -3f - 2.$$

Hence $2u - 3M = L = 9(12) - p - 1$. Thus

$$(33)\qquad 4p = L^2 + 27M^2,\quad L \equiv 1 \pmod 3),$$

$$(34)\qquad 9(12) = p + 1 + L,\quad 9(00) = p - 8 + L,$$

$$(35)\quad 18(01) = 2p - 4 - L + 9M,\quad 18(02) = 2p - 4 - L - 9M.$$

Hence by (30) all nine (ij) are expressed in terms of p, L, M. By the theory of binary quadratic forms, L^2 and M^2 are uniquely determined by (33). The sign of L has been chosen so that congruence (33) holds. But the sign of M depends on the primitive root g employed; see below (93).

10. *Higher congruences.*

THEOREM 1. *If no c_i is divisible by the prime $p = ef + 1$, the number of solutions $x_1, \cdots, x_n$ all prime to p of*

$$(36)\qquad \sum_{i=1}^{n} c_i x_i^e \equiv d \pmod p$$

is e^n times the number of sets of values of $z_1, \cdots, z_n$, each chosen from $0, 1, \cdots, f - 1$, which satisfy

$$(37)\qquad \sum_{i=1}^{n} g^{ez_i + a_i} \equiv d \pmod p,$$

where g is a primitive root of p and $c_i \equiv g^{a_i} \pmod p$.

We may write $x_i \equiv g^{y_i} \pmod p$, $0 \leqq y_i \leqq p - 2$. Divide y_i by f. Then

$y_i = q_i f + z_i$, $0 \leqq z_i \leqq f - 1$, $0 \leqq q_i \leqq e - 1$. The number of solutions of (36) prime to p is the number of sets $y_1, \cdots, y_n$ taken modulo $p - 1$ which satisfy

$$(38) \qquad \sum_{i=1}^{n} g^{a_i + e(q_i f + z_i)} \equiv d \pmod{p}.$$

Since $g^{ef} \equiv 1$, (38) reduces to (37) for each of the e^n sets $q_1, \cdots, q_n$.

Let $n = 2$, $a_1 = a_2 = 0$, $d = 1$. Then (37) is

$$g^{ez_1} + g^{ez_2} \equiv 1, \qquad 1 + g^{ev} \equiv g^{ew} \pmod{p}.$$

THEOREM 2. *The number* * *of solutions prime to* $p = ef + 1$ *of*

$$(39) \qquad x^e + y^e \equiv 1 \ (mod\ p)$$

is $e^2(0, 0)$. *The number of all solutions is* $2e + e^2(0, 0)$.

THEOREM 3. *For* k *and* h *chosen from* $0, \cdots, e - 1$, *the congruence*

$$(40) \qquad 1 + g^k x^e \equiv g^h y^e \ (mod\ p = ef + 1)$$

has exactly $e^2(k, h)$ *solutions if* $h \neq 0$ *and*

$$(41) \qquad k \neq 0 \text{ if } f \text{ even}, \qquad k \neq e/2 \text{ if } f \text{ odd};$$

$e + e^2(k, h)$ *solutions if* $h = 0$ *and* (41), *or if* $h \neq 0$ *and*

$$(42) \qquad k = 0 \text{ if } f \text{ even}, \qquad k = e/2 \text{ if } f \text{ odd};$$

$2e + e^2(k, h)$ *solutions if* $h = 0$ *and* (42).

By (5) and Theorem 1, (40) has exactly $e^2(k, h)$ solutions prime to p. The number of solutions with $x = 0$ is e or 0 according as h is or is not divisible by e. Next, $y = 0$ if and only if

$$-g^k \equiv g^{k + (p-1)/2}$$

is an e-th power, viz., $k + (p - 1)/2$ divisible by e. When f is even, this is true only if k is divisible by e. When f is odd, $\frac{1}{2}(p - 1) = \frac{1}{2}e + e(f - 1)/2$, the condition is $k \equiv \frac{1}{2}e \pmod{e}$.

THEOREM 4. *When* f *is even, the congruence*

$$(43) \qquad 1 + g^k x^e \equiv -g^h y^e \ (mod\ p = ef + 1)$$

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* False result by G. Cornacchia, *Giornale di Matematico*, vol. 47 (1909), pp. 225, 235, 238, 241, etc.

has the same number of solutions as (40). *When f is odd, the number of solutions is $N = e^2(k, h \pm \frac{1}{2}e)$ if $h \neq \frac{1}{2}e$, $k \neq \frac{1}{2}e$; $e + N$ if $h \neq \frac{1}{2}e$, $k = \frac{1}{2}e$, or $h = \frac{1}{2}e$, $k \neq \frac{1}{2}e$; $2e + N$ if $h = k = \frac{1}{2}e$.*

When f is even, there exists an integer w belonging to the exponent $2e$, a divisor of $p - 1 = 2e \cdot f/2$, whence $w^e \equiv -1 \pmod{p}$.

When f is odd, $-g^h \equiv g^H \pmod{p}$, where $H = h - \frac{1}{2}e + e(f+1)/2$, and (43) is equivalent to

$$1 + g^k x^e \equiv g^{h-e/2} Y^e \pmod{p}.$$

Hence we apply Theorem 3 with h replaced by $h - \frac{1}{2}e$. The case $k = h = 0$ gives

THEOREM 5. *If f is odd, the number of solutions of*

$$1 + x^e + y^e \equiv 0 \quad (mod\ p = ef + 1) \tag{44}$$

is $e^2(0, \frac{1}{2}e)$. If f is even, it has the same number of solutions as (37).

THEOREM 6. *If r, s, A are all prime to $p > 2$,*

$$rx^2 + sy^2 = A \quad (mod\ p) \tag{45}$$

has $p - N$ solutions, where $N = +1$ or -1 according as $-rs$ is a quadratic residue or non-residue of p.*

Since the theorem and the congruence are unaltered if we multiply r, s, A by the same integer prime to p, it suffices to prove the theorem for the case $A = -1$. Hence it suffices to prove that

$$1 + rx^2 \equiv ty^2 \pmod{p} \tag{46}$$

has $p - N$ solutions, where $N = +1$ or -1 according as rt is a quadratic residue or non-residue of p.

Since a primitive root g is a quadratic non-residue of p, there are four cases: r, $t = 1$ or g. By Theorem 3 with $e = 2$, the number of solutions when f is odd is

$$2 + 4(0,0) \text{ if } r = t = 1, \quad 4(0,1) \text{ if } r = 1,\ t = g;$$
$$4 + 4(1,0) \text{ if } r = g,\ t = 1; \quad 2 + 4(1,1) \text{ if } r = t = g;$$

* Jordan, *Traité des substitutions* (1870), pp. 156-161; *Comptes Rendus*, vol. 62 (1866), p. 687 (Lebesgue, *ibid.*, p. 868). The case of n variables is proved by induction on n.

but when f is even is $4+4(0,0)$, $2+4(0,1)$, $2+4(1,0)$, $4(1,1)$, in the respective cases. Applying (18) and (19), we obtain the statement below (46).

By Theorem 2 and (34),

$$x^3+y^3\equiv 1 \pmod{p=3f+1} \tag{47}$$

has exactly $p-8+L$ solutions prime to p. In case 2 is a cubic residue of p, $2y^3\equiv 1 \pmod p$ has three roots and (47) has nine solutions prime to p with $x^3\equiv y^3$. The solutions prime to p with $x^3\not\equiv y^3$ fall into sets of $2\times 3\times 3$ (where those of a set have fixed values of x^3 and y^3, also permuted). Hence $p-8+L\equiv 9 \pmod{18}$ and L is even. But if 2 is a cubic non-residue of p, $p-8+L\equiv 0 \pmod{18}$ and L is odd.

THEOREM 7. *Congruence* (47) *has* $p-2+L$ *solutions in all.* 2 *is a cubic residue of* p *if and only if* L *is even and* $p=l^2+27m^2$ *is then solvable.*

11. *Case* $e=4$. Here $p=4f+1$. By (27) with $\beta^2=-1$,

$$\begin{aligned}R(1,1)=(00)-(01)+(02)-(03)-(20)+(21)-(22)+(23)\\+2\beta\{(10)-(11)+(12)-(13)\}.\end{aligned}$$

Case $e=4$, f even. For application to $e=8$, we here write $[ij]$ for the usual (ij). Then $[h,k]=[k,h]$ and

$$[13]=[23]=[12],\ [11]=[03],\ [22]=[02],\ [33]=[01]. \tag{48}$$

Let L_1, L_2, L_3 denote the following equations, from (17):

$$\begin{aligned}&[00]+[01]+[02]+[03]=f-1,\ [01]+[03]+2[12]=f,\\ &[02]+[12]=\tfrac{1}{2}f.\end{aligned} \tag{49}$$

$$\begin{aligned}L_1-L_2-L_3 &: 3[12]=[00]+\tfrac{1}{2}f+1,\\ L_1-2L_2-2L_3 &: [00]-[01]-[02]-[03]-6[12]=-2f-1,\end{aligned}$$

$$R(1,1)=-x+2\beta y,\quad x=2f+1-8[12],\quad y=[01]-[03]. \tag{50}$$

$$p=x^2+4y^2,\quad x\equiv 1 \pmod 4. \tag{51}$$

Here y is two-valued, depending on the choice of the primitive root g; see below (93). We get

$$\begin{aligned}&16[00]=p-11-6x,\quad 16[01]=h+8y,\quad 16[02]=h,\\ &16[03]=h-8y,\quad 16[12]=p+1-2x,\quad h=p-3+2x.\end{aligned} \tag{52}$$

12. *Case* $e=4$, f odd. By the correspondence (16), or direct,

$$(53)\qquad \begin{gathered} 22 = 20 = 00,\ 32 = 13 = 01,\ 12 = 31 = 03, \\ 33 = 23 = 30 = 21 = 11 = 10, \end{gathered}$$

$$(54)\qquad \begin{gathered} 00 + 01 + 02 + 03 = f,\ 01 + 03 + 2(10) = f,\ 00 + 10 = \tfrac{1}{2}(f - 1), \\ R(1,1) = -00 - 01 + 02 - 03 + 2(10) + 2\beta(03 - 01), \end{gathered}$$

Multiply (54) by $-1, 2, 2$ and add. Hence

$$(55)\qquad R(11) = -x + 2\beta y,\ x = 2f - 1 - 8(10),\ y = (03) - (01).$$

Thus (51) holds. All the (ij) are determined; for example

$$(56)\qquad (02) = 3(10) - \tfrac{1}{2}(f - 1),\quad 16(02) = p + 1 - 6x.$$

13. *Case* $e = 5$. For a prime $p = 5f + 1$, f is even. For application to $e = 10$, write $[ij]$ for the usual (ij). By (14),

$$(57)\qquad 44 = 01,\ 33 = 02,\ 22 = 03,\ 11 = 04,\ 34 = 14 = 12,\ 24 = 23 = 13,$$

and $[kh] = [hk]$. The twenty-five $[ij]$ reduce to 00, 01, 02, 03, 04, 12, 13. Here (17) reduce to

$$(58)\qquad \begin{gathered} 00 + 01 + 02 + 03 + 04 = f - 1,\quad 01 + 04 + 2[12] + 13 = f, \\ 02 + 03 + 12 + 2[13] = f. \end{gathered}$$

Let β be a primitive fifth root of unity, whence

$$(59)\qquad \beta^4 + \beta^3 + \beta^2 + \beta + 1 = 0.$$

We eliminate the terms free of β from (27) and obtain

$$R(1,1) = a_1\beta + a_2\beta^2 + a_3\beta^3 + a_4\beta^4,$$

$$(60)\qquad \begin{gathered} a_1 = [02] - [00] + 2[01] - 2[12],\ a_2 = [04] - [00] + 2[02] - 2[13], \\ a_3 = [01] - [00] + 2[03] - 2[13],\ a_4 = [03] - [00] + 2[04] - 2[12]. \end{gathered}$$

By (26) and (27),

$$\begin{gathered} p = a_1^2 + a_2^2 + a_3^2 + a_4^2 + (\beta + \beta^4)B + (\beta^2 + \beta^3)C, \\ B = a_1a_2 + a_2a_3 + a_3a_4,\quad C = a_1a_3 + a_2a_4 + a_1a_4. \end{gathered}$$

Replace $\beta + \beta^4$ by $-1 - \beta^2 - \beta^3$ and note that (59) is irreducible. Hence

$$(61)\qquad p = a_1^2 + a_2^2 + a_3^2 + a_4^2 - B,\quad B = C.$$

Replacing B by $\tfrac{1}{2}(B + C)$, we see that

$$(62)\qquad \begin{gathered}16p = x^2 + 5[a_1 - a_2 - a_3 + a_4]^2 + 10F^2 + 10G^2,\\ -x = a_1 + a_2 + a_3 + a_4, \quad F = a_2 - a_3, \quad G = a_1 - a_4.\end{gathered}$$

By the values of the a_i, we get

$$(63)\qquad \begin{gathered}x = 25\{[12] + [13]\} - 10f - 4, \quad a_1 - a_2 - a_3 + a_4 = 5w,\\ w = [13] - [12],\\ F = 2u - v, \quad G = u + 2v, \quad u = [02] - [03], \quad v = [01] - [04].\end{gathered}$$

Hence

$$(64)\qquad 16p = x^2 + 50u^2 + 50v^2 + 125w^2,$$

and $x \equiv 1 \pmod 5$. Using $B = C$, we find that

$$D = G^2 + 4FG - F^2 = (a_1 + a_4)^2 - (a_2 + a_3)^2 = -5xw.$$

Using $5u = 2F + G$, $5v = -F + 2G$, we find that

$$(65)\qquad \begin{gathered}25(u^2 + 4uv - v^2) = 5D,\\ v^2 - 4uv - u^2 = xw.\end{gathered}$$

By (58) and the value of x,

$$(66)\qquad \begin{gathered}[00] - 3[12] - 3[13] + f + 1 = 0,\\ 25[00] = p - 14 + 3x.\end{gathered}$$

Hence by Theorem 2,

$$(67)\qquad X^5 + Y^5 \equiv 1 \pmod{p = 5f + 1} \text{ has } p - 4 + 3x \text{ solutions.}$$

By (62) and the definition (63) of w, we get

$$(68)\qquad 4a_1, 4a_4 = 5w - x \pm 2G; \quad 4a_2, 4a_3 = -5w - x \pm 2F.$$

THEOREM 8. *There are exactly eight integral simultaneous solutions of* (64) *and* (65). *If* (x, u, v, w) *is one solution, also* $(x, -u, -v, w)$ *and* $(x, \pm v, \mp u, -w)$ *are solutions. The remaining four are derived from these four by changing all signs.*

I. *Elementary proof.* Since 5 is a quadratic residue of a prime $p = 5f + 1$, there are two roots of $s^2 \equiv 5 \pmod p$. We have

$$(64')\qquad 50(u^2 + v^2) \equiv -x^2 - 125w^2 \pmod p.$$

In (65) transpose $4uv$, square and eliminate $u^4 + v^4$ by means of the square of (64′), and multiply by $s^2 \equiv 5$. We get

$$(69)\qquad s(x^2 - 125w^2) \equiv 100(xw + 5uv) \pmod p.$$

From the products of (64′) and (69) by 50 and 10,

$$(70)\quad 2500(u+v)^2 \equiv -1000xw - 50(x^2+125w^2) + 10s(x^2-125w^2).$$

Employ an integer r belonging to the exponent 5 modulo $p=5f+1$. Write $a=r^4-r$, $b=r^3-r^2$. Then $a^2+b^2\equiv -5$, $a^2-b^2\equiv s$, where $s=-r^4+r^3+r^2-r$, $s^2\equiv 5$, and $ab\equiv s$. Define $m=-2a-4b$, $t=4a-2b$. Then

$$(71)\qquad m^2\equiv 10s-50,\quad t^2\equiv -10s-50,\quad mt\equiv -20s.$$

Hence (70) is the square of either

$$(72)\qquad 50(u+v)\equiv mx+5stw \pmod p$$

or the like congruence in which the signs of u and v are both changed. This change is taken care of in the theorem.

Write $2K=t+m$, $2L=t-m$. Then (72) is the sum of

$$(73)\qquad 50u\equiv Kx+5sLw,\quad 50v\equiv -Lx+5sKw \pmod p.$$

The product of (73) agrees with (69). The ambiguity in the determination of u and v is removed as follows. Replace r by r^2 and w by $-w$. Then K, L, s, u, v become $L, -K, -s, -v, u$ respectively. Hence (73) hold either for the given solution (x,u,v,w) or for the new solution $(x,-v,u,-w)$ of (64) and (65).

Let (x,u,v,w) and (x_1,u_1,v_1,w_1) be any integral solutions of (64), (65), (73). Evidently

$$xx_1+50uu_1+50vv_1+125ww_1\equiv 0 \pmod p.$$

Denote the absolute value of the left member by A. By (64),

$$(16p)^2 = A^2+50(xu_1-x_1u)^2+50(xv_1-x_1v)^2+125(xw_1-x_1w)^2$$
$$+2500(uv_1-u_1v)^2+6250(uw_1-u_1w)^2+6250(vw_1-v_1w)^2.$$

Hence $A\leqq 16p$, $6p^2\equiv A^2 \pmod{25}$. By (64), $x\equiv w$, $x_1\equiv w_1 \pmod 2$. Hence $A=2mp$,

$$4m^2\equiv 6 \pmod{25},\quad 2m=5j\pm 1,\quad j\equiv \pm 3 \pmod 5,\quad 2m\equiv \pm 16 \pmod{25}.$$

Hence $A=16p$,

$$xu_1-x_1u=0,\cdots, vw_1-v_1w=0.$$

Since (64) implies $x^2\equiv x_1^2\equiv 1 \pmod 5$, $x\neq 0$, $x_1\neq 0$,

$$u/x = u_1/x_1, \quad v/x = v_1/x_1, \quad w/x = w_1/x_1.$$

Hence (64) gives $x^2 = x_1^2$, whence $u^2 = u_1^2$, etc. This proves * Theorem 8.

Choose a definite one of the eight solutions of (64) and (65). Then the three linear equations (58) and the four linear equations whose left members are x, w, u, v uniquely determine $[0h]$, $h = 0, \cdots, 4$, and $[12]$, $[13]$, and hence determine uniquely all 25 numbers $[i, j]$. This solves the cyclotomic problem for $e = 5$.

II. *Proof of Theorem 8 by algebraic numbers.* Let $p \equiv 1 \pmod 5$. In the field F defined by an imaginary fifth root β of unity, the principal ideal (p) is the product † of four distinct prime ideals each of norm p. Since the class-number of F is 1, every ideal is a principal ideal. Hence

$$(p) = (p_1) \cdots (p_4), \qquad p = Up_1 \cdots p_4,$$

where p_i is a polynomial in β^i with integral coefficients independent of i, and U is a unit. Write $f(\beta)$ for p_1p_2. Then $f(\beta^4) = p_4p_3$. The symmetric function $p_1p_2p_3p_4$ is an integer I. Hence $p = UI$, $U = \pm 1$. Thus $\pm p = f(\beta)f(\beta^{-1})$. The lower sign is excluded by (62). Hence $U = 1$ and

$$(74) \qquad p = f(\beta)f(\beta^4), \qquad f(\beta) = a_1\beta + a_2\beta^2 + a_3\beta^3 + a_4\beta^4.$$

Similarly, $p_1p_3 \cdot p_4p_2$ furnishes a decomposition of p of type (74). But if $g(\beta) = p_1p_4$, then $g(\beta^2) = p_2p_3$ is not the product of $g(\beta^4) = g(\beta)$ by a unit, and we do not obtain a decomposition (74).

The replacement of β by β^3 yields $(p_1p_3p_4p_2)$ and replaces (74) or $p_1p_2 \cdot p_3p_4$ by $p_3p_1 \cdot p_4p_2$ or $f(\beta^3)f(\beta^2)$, and gives rise to the substitution $S = (a_1a_2a_4a_3)$. The replacement of β by β^4 of β^2 gives rise to the square $(a_1a_4)(a_2a_3)$ or cube $(a_1a_3a_4a_2)$ of S.

Now S replaces x, u, v, w by $x, -v, u, -w$. Hence apart from the powers of S, the only decompositions of p into two conjugate factors are

$$p = Vf(\beta) \cdot V^{-1}f(\beta^4),$$

where V is a unit. Every unit of the field F is of the form

$$V = \pm \beta^k J^n, \qquad J = \beta + \beta^4,$$

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* In the much longer proof by G. Hull, *Transactions of the American Mathematical Society*, vol. 34 (1932), pp. 908-937, the sign of ζ in (87) should be changed. His y, z are our $u - v$, $-u - v$. Our u, v, w, x correspond to C, D, $A - B$, $\frac{1}{8}\{4p - 16 - 25(A + B)\}$ of W. Burnside, *Proceedings of the London Mathematical Society*, (2), vol. 14 (1915), pp. 251-259.

† Kummer. Cf. Hilbert's Report, *Jahresbericht der Mathematischen Vereinigung*, Bd. 4 (1894-1895), pp. 328-329.

where k and n are integers. The condition for $V^{-1}=V(\beta^4)$ is $J^{-n}=J^n$. But

$$J^2+J=1, \qquad 2J=-1\pm 5^{1/2}, \qquad |J|\neq 1.$$

Hence $n=0$. If we change the sign of each factor in (74), we change the signs of each a_i and hence of x, u, v, w. We have now accounted by the eight solutions in Theorem 6.

It remains only to consider

$$p=\beta^k f(\beta)\cdot\beta^{4k}f(\beta^4).$$

In view of our examination of the effect of replacing β by β^k, it suffices to treat the case $k=1$. For f in (74),

$$\beta f=A_1\beta+\cdots+A_4\beta^4,\ A_1=-a_4,\ A_2=a_1-a_4,\ A_3=a_2-a_4,\ A_4=a_3-a_4.$$

Let V denote the function obtained from v by replacing a_i by A_i. By the analogue of $5v=-F+2G$, we get

$$5V=-(A_2-A_3)+2(A_1-A_4)=a_2-a_1-2a_3,$$
$$20V=2x+2(3F-G)=2x+10u-10v, \quad x\equiv 0 \pmod 5,$$

contrary to $x^2\equiv 1$ by (64). Expressed otherwise, if a_i are integral solutions of (61) to which correspond integral solutions of (64) and (65), although the A_i evidently satisfy (61), the corresponding solutions $X,\cdots,W$ of (64) and (65) are not integers.

Example. $p=11$, $a_1=0$, $a_2=-1$, $a_3=-2$, $a_4=2$. Then $x=w=1$, $u=0$, $v=-1$; $A_1=A_2=-2$, $A_3=-3$, $A_4=-4$; $X=11$, $U=4/5$, $V=3/5$, $W=-1/5$.

14. *Subdivision of periods.* Let d be any divisor of e and write $E=e/d$. Then $(p-1)/E=df$. Replacing e, f by E, df in (1), we see that the E periods are

$$Y_k=\sum_{t=0}^{df-1}R^{g^{Et+k}} \qquad (k=0,\cdots,E-1).$$

The values $j, d+j,\cdots,(f-1)d+j$ of t give the terms of η_{k+jE} in (1). Hence

$$Y_k=\sum_{j=0}^{d-1}\eta_{k+jE}. \tag{75}$$

Take $d=2$. Then $e=2E$ and

$$\begin{aligned} Y_k&=\eta_k+\eta_{k+E},\\ Y_0Y_k&=(\eta_0+\eta_E)(\eta_k+\eta_{k+E}),\\ \eta_0\eta_k&=fn_k+\sum_{h=0}^{E-1}(k,h)\eta_h+\sum_{H=0}^{E-1}(k,E+H)\eta_{E+H}, \end{aligned} \tag{76}$$

from which we get $\eta_0\eta_{k+E}$ by replacing k by $k+E$. Similarly

$$\eta_E\eta_{E+k} = fn_k + \sum_{h=0}^{E-1}(k,h)\eta_{E+h} + \sum_{H=0}^{E-1}(k, E+H)\eta_H,$$

by (7). Replacing m by k and k by $E-k$ in (7), we get

$$\eta_k\eta_E - fn_{E-k} = \sum_{h=E-k}^{2E-1-k}(E-k,h)\eta_{k+h} + \sum_{h=0}^{E-1-k} + \sum_{h=2E-k}^{2E-1}.$$

In the first sum, take $k+h=E+H$; we get

$$\sum_{H=0}^{E-1}(E-k, E+H-k)\eta_{E+H}.$$

In the second and third sums, take $k+h=H$. In the last case we may drop $2E$ from the subscripts of η. Combining, we get

$$\sum_{H=0}^{E-1}(E-k, H-k)\eta_H.$$

The total sum must be equal to (6) for Y periods:

$$Y_0Y_k = \sum_{h=0}^{E-1}(k,h)_E Y_h + 2fN_k, \qquad Y_h = \eta_h + n_{h+E}.$$

By the coefficients of η_h,

(77) $\quad (k,h)_E = (k,h) + (k+E,h) + (k,E+h) + (E-k, h-k).$

By way of check, we may verify that the coefficient of η_{h+E} in the total sum is also (77). By (14) and (15),

(78) $\quad (00)_E = (00) + 3(0E),\ f$ even; $\quad (00)_E = 3(00) + (0E),\ f$ odd.

In (22) for $e=2E$, $m=2M$, take $j=J+E$ in the terms with $j=E,\cdots,2E-1$. By (76), we get

$$F(\beta^{2M}) = \sum_{j=0}^{E-1}\beta^{2Mj}Y_j.$$

Now $B=\beta^2$ is a primitive E-th root of unity. Let $\phi(B^m)$ denote the function derived from $F(\beta^m)$ in (22) by replacing e by E, β by B, and η by Y. Hence $F(\beta^{2M}) = \phi(B^m)$. Applying (26) also for ϕ, we get

(79) $\quad R(2r,2s)_e = \{R(r,s)_E$ with β replaced by $\beta^2\}.$

15. *Jacobi's Theorem.** If $g^m \equiv 2 \pmod p$, function (21) has the property

$$F(-1)F(\alpha^2) = \alpha^{2m}F(\alpha)F(-\alpha). \tag{80}$$

For i fixed, the coefficient of α^i in $F(\alpha)F(-\alpha)$ is

$$\sum_{j=0}^{p-2}(-1)^j R^c, \qquad c \equiv g^{i-j} + g^j \pmod p. \tag{81}$$

Hence this is the coefficient of α^{2m+i} in the second member of (80). If i is odd, the sum (81) is zero since $j = J$ and $j = i - J$ give rise to the same value of c modulo p, while one of J, $i - J$ is even and the other is odd.

Henceforth, let i be even, $i = 2t$. Thus we seek the coefficient of α^{2m+2t} in $F(\alpha^2)$. It is obtained by replacing α by α^2 in the terms of $F(\alpha)$ in (21) having $k = m + t$ and $k = m + t + \frac{1}{2}(p-1)$. Hence the coefficient of α^{2m+2t} in $F(-1)F(\alpha^2)$ is

$$F(-1)(R^{g^{m+t}} + R^{-g^{m+t}}),$$

or, by use of $g^m \equiv 2$,

$$\sum_{k=0}^{p-2}(-1)^k R^{g^k+2g^t} + \sum_{k=0}^{p-2}(-1)^k R^{g^k-2g^t}. \tag{82}$$

First, let $J \not\equiv t\ [\mathrm{mod}\ \frac{1}{2}(p-1)]$. Then J and $i - J$ are values of j incongruent modulo $p - 1$, leading to the same c in (81), and the coefficient of R^c is $2(-1)^J$. The term R^c occurs in the first sum (82) for $g^k \equiv g^{-J}(g^t - g^J)^2$, and occurs in the second sum for $g^k \equiv g^{-J}(g^t + g^J)^2$. In each case g^k and g^J are both quadratic residues or both non-residues of p, whence $k \equiv J \pmod 2$. Thus the coefficient of R^c in (82) is $2(-1)^J$.

Second, let $J \equiv t\ [\mathrm{mod}\ \frac{1}{2}(p-1)]$. Then $g^J \equiv \pm g^t$, $c \equiv \pm 2g^t \pmod p$. Now only one of the two sums (82) yields a term R^c, the second when $g^k \equiv 4g^t$ and the upper sign holds, but the first when $g^k \equiv -4g^t$ and the lower sign holds. In both cases, $g^k \equiv 4g^J \pmod p$, $k \equiv J \pmod 2$, whence $(-1)^J$ is the coefficient of R^c in both (81) and (82).

It remains to consider exponents c that do not occur in (81). Write z for g^j. Then $g^{2t}/z + z \equiv c \pmod p$ has no root z. Hence $c^2 - 4g^{2t}$ is a non-residue of p. Hence for one of the congruences

$$c - 2g^t \equiv g^k, \qquad c + 2g^t \equiv g^k \pmod p,$$

the solution g^k is a residue and for the other a non-residue. Thus x^c occurs in one of the sums (82) with the coefficient $+1$ and in the other with -1.

* Stated without proof in *Journal für Mathematatik*, Bd. 30 (1846), p. 167. The present proof was recently obtained by H. H. Mitchell.

Since we have found that (81) and (82) have the same coefficients of R^c for every c, we have proved (80).

16. *The reduced* $R(m,n)$. By (25) and (26)

$$R(n,m)=R(m,n)=(-1)^{nf}R(-m-n,n), \tag{83}$$

when no one of m, n, $m+n$ is divisible by e.

When β is replaced by a new primitive e-th root β^j of unity (j prime to e), $R(m,n)$ becomes $R(jm,jn)$. The latter is called a conjugate of the former. The relation obtained from (28) by this replacement evidently yields the same decomposition of p into integers that (28) itself yields.

When we retain only one of a set of conjugate R's and discard duplicates by (83), we obtain a set of reduced R's.

Examples of complete sets of reduced R's:

$$e=6:\ R(1,1);\ R(1,2),\ R(2,2).$$
$$e=8:\ R(1,1),\ R(1,3),\ R(1,5),\ R(2,2).$$

17. Theorem 9. *When $e=6$, the 36 cyclotomic constants (k,h) depend solely upon the decomposition A^2+3B^2 of the prime $p=6f+1$.*

By (83), $R(11)=(-1)^fR(14)$. Employ the values of $R(14)$ and $R(12)$ from (26) and apply (80) for $\alpha=\beta^2$; we get the first of

$$R(11)=(-1)^f\beta^{4m}R(12),\quad R(22)=\beta^{2m}R(12), \tag{84}$$

the second of which follows from (80) for $\alpha=\beta$. Here

$$\beta^3=-1,\quad 2\beta=1+(-3)^{1/2},\quad 2\beta^2=-1+(-3)^{1/2}.$$

By (79) and (32), we get the first of

$$2R(22)=L+3M(-3)^{1/2},\quad R(12)=-A+B(-3)^{1/2}, \tag{85}$$
$$2R(11)=E+F(-3)^{1/2}.$$

18. *Case $e=6$, f even.* Since our results will be needed for $e=12$, we shall here write $[ij]$ for the usual (ij). Then

$$\begin{aligned}&[kh]=[hk],\quad [01]=[55],\quad 02=44,\quad 03=33,\quad 04=22,\\ &05=11,\quad 12=15=45,\quad 13=25=34,\quad 14=23=35.\end{aligned} \tag{86}$$

We retain the first one in each equation and $[00]$, $[24]$. Then (17) reduce to

$$\begin{aligned}&00+01+02+03+04+05=f-1,\\ &01+05+2[12]+13+14=f,\\ &02+04+12+13+14+24=f,\quad 03+13+14=\tfrac{1}{2}f.\end{aligned} \tag{87}$$

Multiply these by $-1, 1, 1, -2$, and add; we get

$$(88)\qquad [24]-[00]-3[03]+3[12]=1.$$

$$(89)\qquad \begin{aligned} &A=2[24]-2[00]-1,\quad B=[01]-[05]-[13]+[14],\\ &E=2[00]-6[03]-3[12]+7[24],\\ &F=[01]+2[13]-3[04]+3[02]-[05]-2[14].\end{aligned}$$

Thus $A\equiv 1 \pmod 6$. By (32), (78); (31_2), (77); and (33)

$$(90)\qquad \begin{aligned} &9[00]+27[03]=p-8+L,\quad 4p=L^2+27M^2,\quad L\equiv 1 \pmod 3,\\ &M=[01]+2[13]+[04]-[02]-2[14]-[05].\end{aligned}$$

I. Let 2 be a cubic residue of p. Then $\beta^{2m}=1$. By (84), (85), $L=E=-2A$, $F=2B=3M$. Then (87)-(90) give

$$(91)\qquad \begin{aligned} &36[00]=p-17-20A,\quad 36[03]=\pi,\quad 36[12]=p+1-2A,\\ &36[01]=\pi+18B,\quad 36[05]=\pi-18B,\quad 36[02]=\pi+6B,\\ &36[04]=\pi-6B,\quad [13]=[14]=[24]=[12],\quad \pi=p-5+4A.\end{aligned}$$

II. In $g^m\equiv 2 \pmod p$, let $m\equiv 2$ or $5 \pmod 6$. Then

$$E=A-3B,\quad F=-A-B,\quad L=A+3B,\quad 3M=A-B,$$

$$(92)\qquad \begin{aligned} &36[00]=p-17-8A-6B, 36[03]=\pi+6B, 36[24]=p+1+10A-6B,\\ &36[12]=36[14]=p+1-2A+6B, 36[05]=\pi-12B,\ [04]=[01]=[03],\\ &36[13]=p+1-2A-12B, 36[02]=p-5-8A, \pi=p-5+4A.\end{aligned}$$

III. Let $m\equiv 1$ or $4 \pmod 6$. Then $E=A+3B$, $F=A-B$, $L=A-3B$, $3M=-A-B$, and

$$(93)\qquad \begin{aligned} &36[00]=p-17-8A+6B, 36[03]=36[02]=36[05]=\pi-6B,\\ &36[12]=36[13]=p+1-2A-6B, 36[24]=p+1+10A+6B,\\ &36[14]=p+1-2A+12B,\ 36[01]=\pi+12B,\ 36[04]=p-5-8A.\end{aligned}$$

We may deduce case III from II as follows.

For any e, f, replace g by a new primitive root g^r of p, where r is prime to $p-1$. Then η_k in (1) becomes η_{rk} since rt ranges with t over a complete set of residues modulo f. By (6), (k, h) becomes (rk, rh).

Let $r'r\equiv 1 \pmod e$. Since $rj=J$ ranges with j over a complete set of residues modulo e, $F(\beta^m)$ in (22) becomes

$$\sum_{J=0}^{e-1}\beta^{mr'J}\eta_J=F(\beta^{mr'}).$$

By (26), $R(m, n)$ becomes $R(mr', nr')$.

THEOREM 10. *When g is replaced by a new primitive root g^r, $R(m,n)$ becomes $R(mr', nr')$, where $r'r \equiv 1 \pmod e$. The effect on any F or R is to replace β by $\beta^{r'}$.*

For our case $e=6$, f even, take $r \equiv 5 \pmod 6$. The replacement of β by β^{-1} is equivalent to changing the sign of $(-3)^{1/2}$. Hence by (85), A, E, L, 00, 03, 12 and 24 are unaltered, while B, F, M are changed in sign. But 01 and 05, 02 and 04, 13 and 14 are interchanged. Then (92) become (93) and conversely.

19. *Case $e=6$, f odd.* We retain 03 and the first one in each equation

$$(94)\quad \begin{aligned} &00=30=33,\quad 01=25=43,\quad 02=14=53,\quad 21=45,\\ &04=13=52,\quad 05=23=41,\quad 10=22=31=34=40=55,\\ &11=20=32=35=44=50,\quad 15=12=24=42=51=54;\end{aligned}$$

$$(95)\quad \begin{aligned} &00+01+02+03+04+05=f,\quad 02+04+10+11+2(15)=f,\\ &01+05+10+11+15+21=f,\quad 00+10+11=\tfrac{1}{2}(f-1);\end{aligned}$$

$$(21)-(03)-3(00)+3(15)=1,$$

$$(96)\quad \begin{aligned} &L=27(00)+9(03)-p+8,\quad M=01+04+2(10)-02-05-2(11),\\ &A=2(03)-2(21)\equiv 4 \pmod 6,\quad B=10-11+02-04,\\ &E=2(03)-6(00)-3(12)+7(21)-2,\\ &F=04-3(01)+2(10)-02+3(05)-2(11).\end{aligned}$$

While L and M are the same functions of A, B as in I-III, E and F are the negatives of their former functions of A, B.

I. $2=$ cubic residue of p. For $t=p+1-2A$,

$$(97)\quad 36(00)=p-11-8A,\quad 36(03)=t+18A,\quad 36(15)=36(21)=t.$$

II. In $g^m \equiv 2 \pmod p$, let $m \equiv 2$ or $5 \pmod 6$, $q=p+1+A+3B$. Then

$$(98)\quad \begin{aligned} &36(00)=p-11-2A,\quad 36(03)=q+9A+9B.\\ &36(15)=q+3A-3B,\quad 36(21)=q-9A+9B.\end{aligned}$$

III. If $m \equiv 1$ or $4 \pmod 6$, change the sign of B in II.

20. THEOREM 11. *When $e=8$, the 64 cyclotomic constants (k,h) depend solely upon the decompositions $p=x^2+4y^2$ and $p=a^2+2b^2$, $x \equiv a \equiv 1 \pmod 4$.*

Here $p=8f+1$, $\beta^4=-1$, $(\beta+\beta^3)^2=-2$. By (26) and (80) for $\alpha=\beta^3$, we get $R(16)=\beta^{6m}R(13)$. Next, employ (80) for $\alpha=\beta$ and divide by $F(\beta^6)$. Hence $R(24)=\beta^{2m}R(15)$. Applying (83), we get

$$(99)\quad R(22)=\beta^{2m}R(15),\ R(11)=(-1)^f\beta^{6m}R(13),\ g^m\equiv 2 \pmod{p}.$$

21. *Case $e=8$, f even.* By (14), $(hk)=(kh)$ and

$$\begin{gathered}11=07,\ 17=12,\ 22=06,\ 23=16,\ 26=24,\ 27=13,\ 33=05,\\ 34=15,\ 35=25,\ 36=25,\ 37=14,\ 44=04,\ 45=14,\ 46=24,\\ 47=15,\ 55=03,\ 56=13,\ 57=16,\ 66=02,\ 67=12,\ 77=01.\end{gathered}$$

Hence each of the sixty-four (ij) is equal to one of the fifteen:

$$(100)\qquad (0h),\ h=0,\cdots,7;\ (1h),\ h=2,\cdots,6;\ (24),\ (25).$$

Write $[ij]$ for $(ij)_4$. Their values are given by (52). Then by (77),

$$(101)\quad \begin{aligned}&(00)=[00]-3(04),\quad (01)=[01]-(05)-2(14),\\ &(02)=[02]-(06)-2(24),\quad (03)=[03]-(07)-2(15),\\ &(12)=[12]-(13)-(16)-(25).\end{aligned}$$

Eliminating the left members from (17), we get

$$(102)\quad \begin{aligned}&[00]+[01]+[02]+[03]=2f-1,\quad [02]+[12]=f,\\ &[01]+2[12]+[03]=2f,\\ &(07)=[03]+(05)+(13)+(14)-(15)+(16)+2(25)-f,\\ &(04)+(14)+(15)+(24)=\tfrac{1}{2}f,\end{aligned}$$

the first three of which are (49) with f replaced by $2f$. By (79), (50), (51),

$$(103)\qquad R(22)=-x+2\beta^2 y,\quad p=x^2+4y^2,\quad x\equiv 1 \pmod{4}.$$

From $R(mn)$ in (27) we eliminate the left members of (101) and (07) by (102), and get

$$\begin{aligned}&R(13)=-a+b(\beta+\beta^3),\quad p=a^2+2b^2,\\ &\quad -a=[00]-[01]+[02]-[03]-4(04)+4(14)+4(15)-4(24),\\ &\quad b=[01]-[03]-4(05)-4(13)-4(14)-4(25)+2f.\end{aligned}$$

$$\begin{aligned}&R(15)=A+\beta^2B,\quad p=A^2+B^2,\\ &\quad A=[00]-[02]-2[12]+4\{-(04)+(13)+(16)+(24)\},\\ &\quad B=[01]+2[02]-[03]+4\{-(06)-(14)+(15)-(24)\}.\end{aligned}$$

$$R(11) = A_0 + 2A_1\beta + A_2\beta^2 + 2A_3\beta^3,$$
$$A_0 = [00] - [02] + 2[12] + 4\{-(04) - (13) - (16) + (24)\},$$
$$A_1 = [01] - [12] + 2\{-(05) - (14) + (13) + (25)\},$$
$$A_2 = -[01] + 2[02] + [03] + 4\{-(06) + (14) - (15) - (24)\},$$
$$A_3 = -[03] + [12] - 2(05) - 2(13) - 2(14) - 4(16) - 6(25) + 2f.$$

All the (ij) are uniquely determined by our linear equations and (52). By the first and last of (102), we get

$$(104) \qquad 4[00] - 16(04) = A + A_0 - a - 1.$$

This with the first of (101) yield (00) and (04). Other simple relations are

$$(105) \qquad A_1 - A_3 = 4\{(25) - (12)\}, \quad A_2 + B = 4\{(02) - (06)\}.$$

Since 2 is a quadratic residue of $p = 8f + 1$, there are two cases.

I. Let 2 be a biquadratic residue of p, whence m is a multiple of 4 and $\beta^{2m} = +1$. Then (99) gives

$$(106) \qquad A = -x, \quad B = 2y, \quad A_0 = -a, \quad 2A_1 = 2A_3 = b, \quad A_2 = 0.$$

Then (52), (104) and (101_1) give

$$(107) \quad 64(00) = p - 23 - 18x - 24a, \quad 64(04) = p - 7 - 2x + 8a.$$

II. Let 2 be a biquadratic non-residue of p, whence m is the double of an odd integer, and $\beta^{2m} = -1$. Then by (99),

$$(108) \quad A = x, \quad B = -2y, \quad A_0 = a, \quad 2A_1 = 2A_3 = -b, \quad A_2 = 0.$$

$$(109) \qquad 64(00) = p - 23 + 6x, \quad 64(04) = p - 7 - 10x.$$

Examples. If $p = 17$, $(02) = (15) = (16) = 1$; the others of (100) are zero.

p	00	01	02	03	04	05	06	07
97	2	2	0	2	0	2	1	2
113	0	0	0	2	3	2	2	4
257	9	2	2	6	2	0	6	4

p	12	13	14	15	16	24	25
97	1	0	2	1	3	3	1
113	1	3	1	1	3	2	1
257	5	3	5	5	3	4	5

22. *Case* $e = 8$, f *odd*. Here

$14 = 05,\ 13 = 16,\ 15 = 03,\ 22 = 20,\ 23 = 17,\ 24 = 06,\ 25 = 16,\ 26 = 02,$
$27 = 12,\ 30 = 11,\ 31 = 32 = 21,\ 33 = 10,\ 34 = 07,\ 35 = 17,\ 36 = 12,$
$37 = 01,\ 40 = 00,\ 41 = 10,\ 42 = 20,\ 43 = 11,\ 44 = 00,\ 45 = 10,\ 46 = 20,$
$47 = 11,\ 50 = 10,\ 51 = 07,\ 52 = 17,\ 53 = 12,\ 54 = 01,\ 55 = 11,\ 56 = 21,$
$57 = 21,\ 60 = 20,\ 61 = 17,\ 62 = 06,\ 63 = 16,\ 64 = 02,\ 65 = 12,\ 66 = 20,$
$67 = 21,\ 70 = 11,\ 71 = 12,\ 72 = 16,\ 73 = 05,\ 74 = 03,\ 75 = 16,\ 76 = 17,$
$77 = 10.$

Write $[ij]$ for $(ij)_4$, j taken modulo 4. By (77),

$$\begin{aligned} &(04) = [00] - 3(00), \quad (05) = [01] - (01) - 2(10), \\ (110)\qquad &(06) = [02] - (02) - 2(20), \quad (07) = [03] - (03) - 2(11), \\ &(16) = [12] - (12) - (17) - (21). \end{aligned}$$

Eliminate the left members from (17); we get the first three in (102) and

$$(111)\qquad \begin{aligned} &03 = [03] + 01 + 10 - 11 + 12 + 17 + 2(21) - f, \\ &00 + 10 + 11 + 20 = \tfrac{1}{2}(f - 1). \end{aligned}$$

The formulas for $a, \cdots, A_3$ are derived from those for f even by replacing (k, h) by $(k, h + 4)$, from (16), where entries $\geqq 8$ are to be reduced modulo 8. The $[ij]$ are unaltered. We change the sign of a, so that the new a shall be $\equiv 1 \pmod 4$ for f odd or even. As before,

$$(112)\qquad 4[00] - 16(00) = A + A_0 + a + 1.$$

I. Let 2 be a biquadratic residue of p. Then

$$(113)\quad A = -x, \quad B = 2y, \quad A_0 = -a, \quad 2A_1 = 2A_3 = -b, \quad A_2 = 0,$$

$$(114)\qquad 64(00) = p - 15 - 2x, \quad 64(04) = p + 1 - 18x.$$

II. Let 2 be a biquadratic non-residue of p. Then the second members of (113) are to be changed in sign. Thus

$$(115)\quad 64(00) = p - 15 - 10x - 8a, \quad 64(04) = p + 1 + 6x + 24a.$$

23. *Case* $e = 10$. Then $\beta^5 = -1$. By (80) with $\alpha = \beta^2$,

$$(116)\qquad F(\beta^5)F(\beta^4) = \beta^{4m}F(\beta^2)F(\beta^7).$$

Divide by $F(\beta^9)$ and apply (26). Thus $R(45) = \beta^{4m}R(27)$. By (83),

$$R(45) = R(14), \quad R(27) = R(12), \quad R(14) = \beta^{4m}R(12).$$

By (80) with $\alpha = \beta^4$, $R(18) = \beta^{8m}R(14)$. Thus

$$R(18) = \beta^{2m}R(27), \quad F(\beta)F(\beta^8) = \beta^{2m}F(\beta^2)F(\beta^7).$$

Hence by (116), $\beta^{2m}F(\beta)F(\beta^8) = F(\beta^5)F(\beta^4)$. Multiplication by $F(\beta^4)/F(\beta^5)F(\beta^8)$ yields $\beta^{2m}R(14) = R(44)$. By (83), $R(11) = (-1)^f R(18)$. Hence

$$(117) \quad R(14) = \beta^{8m}R(44), \; R(12) = \beta^{4m}R(44), \; R(11) = (-1)^f\beta^{6m}R(44).$$

These four R's are the only reduced ones. By (79), (62) and the remark below (83),

$$(118) \qquad R(44) = -a_4\beta + a_3\beta^2 - a_2\beta^3 + a_1\beta^4.$$

We shall employ the notations

$$(119) \qquad \begin{aligned} R(11) &= b_1\beta + b_2\beta^2 + b_3\beta^3 + b_4\beta^4, \\ R(12) &= d_1\beta + \cdots, R(14) = c_1\beta + \cdots + c_4\beta^4. \end{aligned}$$

Since p in (62) is the product of (60) by its conjugate, by changing β to $-\beta$, we obtain the present analogue of (62) by changing the signs of a_1 and a_3. Just as $(00)_5$ is determined by (66) from $-x = \Sigma a_i$, we shall find here that (00) and (05) are determined by

$$(120) \quad \rho(11) = -b_1 + b_2 - b_3 + b_4, \quad \rho(12) = -d_1 + d_2 - d_3 + d_4,$$
$$\rho(14) = -c_1 + c_2 - c_3 + c_4.$$

I. $m \equiv 0 \pmod 5$. By (117)-(120)

$$\rho(11) = (-1)^f(a_1 + a_2 + a_3 + a_4), \quad \rho(12) = \rho(14) = a_1 + a_2 + a_3 + a_4.$$

II. $2m \equiv 2 \pmod{10}$. Eliminate constant terms by

$$\beta^4 - \beta^3 + \beta^2 - \beta + 1 = 0.$$

Then

$$\rho(11) = (-1)^f(a_2 + a_3 + a_4 - 4a_1), \quad \rho(12) = a_1 + a_2 + a_3 - 4a_4,$$
$$\rho(14) = a_1 + a_2 + a_4 - 4a_3.$$

III. $2m \equiv 4 \pmod{10}$. $\rho(11) = (-1)^f(a_1 + a_3 + a_4 - 4a_2)$,

$$\rho(12) = a_1 + a_2 + a_4 - 4a_3, \quad \rho(14) = a_2 + a_3 + a_4 - 4a_1.$$

IV. $2m \equiv 6 \pmod{10}$. $\rho(11) = (-1)^f(a_1 + a_2 + a_4 - 4a_3)$,

$$\rho(12) = a_1 + a_3 + a_4 - 4a_2, \quad \rho(14) = a_1 + a_2 + a_3 - 4a_4.$$

V. $2m \equiv 8 \pmod{10}$. $\rho(11) = (-1)^f(a_1 + a_2 + a_3 - 4a_4)$,

$\rho(12) = a_2 + a_3 + a_4 - 4a_1$, $\rho(14) = a_1 + a_3 + a_4 - 4a_2$.

24. *Case* $e = 10$, *f even.* We have $(h, k) = (k, h)$ and

$11 = 09$, $19 = 12$, $22 = 08$, $23 = 18$, $28 = 24$, $29 = 13$, $33 = 07$, $34 = 17$, $35 = 27$, $37 = 36$, $38 = 25$, $39 = 14$, $44 = 06$, $45 = 16$, $46 = 26$, $47 = 36$, $48 = 26$, $49 = 15$, $55 = 05$, $56 = 15$, $57 = 25$, $58 = 27$, $59 = 16$, $66 = 04$, $67 = 14$, $68 = 24$, $69 = 17$, $77 = 03$, $78 = 13$, $79 = 18$, $88 = 02$, $89 = 12$, $99 = 01$.

Denote $(kh)_5$ by $[k, h]$ as in § 13. By (77),

$$\begin{aligned}(121)\qquad &(00) = [00] - 3(05), \quad (01) = [01] - (06) - 2(15),\\ &(02) = [02] - (07) - 2(25), \quad (03) = [03] - (08) - 2(27),\\ &(04) = [04] - (09) - 2(16), \quad (12) = [12] - (14) - (17) - (26),\\ &(13) = [13] - (18) - (24) - (36).\end{aligned}$$

The linear relations (17) reduce to (58) with f replaced by $2f$, and

$$\begin{aligned}(122)\qquad (08) &= [03] + [13] + (07) + (14) + (17) - (24) + (25)\\ &\qquad - (27) + (36) - f,\\ (09) &= [04] + (06) + (14) + (15) - (16) + (17) + (24)\\ &\qquad + 2(26) + (36) - f,\end{aligned}$$

$$(123)\qquad (05) + (15) + (16) + (25) + (27) = \tfrac{1}{2}f.$$

In (119) we have

$$\begin{aligned}b_1 &= 00 - 02 - 05 - 07 + 2\{01 - 06 + 12 - 14 - 17 - 2(18) + 25\\ &\qquad + 26 + 2(36)\},\\ b_2 &= 04 - 00 + 05 + 09 + 2\{02 - 07 - 2(12) + 13 + 2(14) - 16 + 18\\ &\qquad - 24 - 36\},\\ b_3 &= 00 - 01 - 05 - 06 + 2\{03 - 08 + 2(12) - 13 + 15 - 2(17) - 18\\ &\qquad + 24 + 36\},\\ b_4 &= 03 - 00 + 05 + 08 + 2\{04 - 09 - 12 + 2(13) + 14 + 17 - 26\\ &\qquad - 27 - 2(36)\}.\end{aligned}$$

Eliminate the left members of (121) and (122); subtract (17) for $k = 0$; and add the double of (123); we get

$$\begin{aligned}(124)\qquad \rho(11) &= 20(05) - 1 - 5[00] + 2\{-[01] + [02] + [03] - [04]\\ &\qquad - 6[12] + 6[13]\} + 20\{(14) + (17) - (24) - (36)\}.\end{aligned}$$

Write $z_j = \Sigma(-1)^h(jh)$, $h = 0, \cdots, 9$. Then in R_{14},

$$(125)\quad \begin{aligned} &c_1 = z_0 + z_1 - z_4, \quad c_2 = -z_0 + z_2 + z_3, \quad c_3 = z_0 - z_2 + z_3, \\ &c_4 = -z_0 + z_1 + z_4, \\ &\rho(14) = -4z_0 + 2z_2 + 2z_4 = -4[00] + 4[01] - 2[02] - 2[03] \\ &\qquad + 4[04] - 2[12] - 8[13] - 2f + 20\{(05)+(24)+(26)\}, \end{aligned}$$

after adding the product of (123) by 4. Next,

$$R(12) = t_0 + t_1\beta + \cdots + t_4\beta^4,$$

where

$$\begin{aligned} t_0 &= [00] - 2[12] - 4(05) + 2(14) + 2(17) + 2(24) + 4(26) - 2(36), \\ t_1 &= [01] - 2(06) - 2(08) - 2(15) + 2(17) - 2(26) + 2(27), \\ t_2 &= [02] + 2(06) - 2(07) - 2(15) - 2(18) + 2(24) - 2(25), \\ t_3 &= [03] - 2[04] + 2[13] - 2(08) + 2(09) + 6(16) - 2(18) - 4(24) \\ &\qquad - 2(27) - 2(36), \\ t_4 &= 2[02] + [04] - 2(07) - 2(09) - 2(14) - 2(16) - 6(25) + 2(26). \\ d_1 &= t_1 + t_0, \quad d_2 = t_2 - t_0, \quad d_3 = t_3 + t_0, \quad d_4 = t_4 - t_0. \end{aligned}$$

$$(126)\quad \begin{aligned} \rho(12) &= -4t_0 - t_1 + t_2 - t_3 + t_4 \\ &= -4[00] - [01] + 3[02] + 3[03] - [04] + 8[12] + 2[13] \\ &\qquad - 2f + 20(05) - 20(26) - 10\{(14)+(17)+(24)-(36)\}, \end{aligned}$$

after adding the product of (123) by 4. By (58) we find that

$$(127)\qquad \rho(11) + 2\rho(12) + 2\rho(14) = 100(05) - 25[00] - 5.$$

This with (121_1), (66) and (68) determine (05) and (00).

I. $m \equiv 0 \pmod 5$. $100(05) = p - 9 - 2x$, $100(00) = p - 29 + 18x$.

II. $2m \equiv 2 \pmod{10}$. $400(05) = 4p - 36 + 17x + 50u - 25w$,
$400(00) = 4p - 116 - 3x - 150u + 75w$.

III. $2m \equiv 4 \pmod{10}$. $400(05) = 4p - 36 + 17x - 50v + 25w$,
$400(00) = 4p - 116 - 3x + 150v - 75w$.

IV. $2m \equiv 6 \pmod{10}$. Change the sign of v in III.

V. $2m \equiv 8 \pmod{10}$. Change the sign of u in II.

25. *Case* $e = 10$, f *odd*. By means of the correspondence (16), which leaves the $[ij]$ unaltered, we may deduce from the results for f even the equalities between the (ij), and the analogues to (121), (122), b_i, c_i, t_i. But (123) is here replaced by

(128) $(00)+(10)+(11)+(20)+(22)=\frac{1}{2}(f-1).$

The present $\rho(11)$ is therefore derived from (124) by replacing -1 by $+1$. But for $k=2$ or 4, the present $\rho(1k)$ is obtained by subtracting 2 from the former $\rho(1k)$. Hence

(129) $\rho(11)-2\rho(12)-2\rho(14)=100(00)-25[00]+5.$

I. $m\equiv 0 \pmod 5$. $100(00)=p-19+8x,\quad 100(05)=p+1-12x.$

II. $2m\equiv 2 \pmod{10}$. $400(00)=4p-76+7x-50u+25w,$
$400(05)=4p+4+27x+150u-75w.$

III. $2m\equiv 4 \pmod{10}$. $400(00)=4p-76+7x+50v-25w,$
$400(05)=4p+4+27x-150v+75w.$

IV. $2m\equiv 6 \pmod{10}$. Change the sign of v in III.

V. $2m\equiv 8 \pmod{10}$. Change the sign of u in II.

26. THEOREM 12. *When $e=12$, the 144 cyclotomic constants (k,h) depend solely upon the decompositions $p=x^2+4y^2$ and $p=A^2+3B^2$ of the prime $p=12f+1$, where $x\equiv 1$ (mod 4), $A\equiv 1$ (mod 6).*

As the reduced R's we may take $R(1k)$, $k=1, 2, 3, 5, 7$, $R(22)$, $R(24)$, $R(33)$, $R(44)$. By (80) with $\alpha=\beta$, β^2 or β^5,

$$R(26)=\beta^{2m}R(17),\quad R(46)=\beta^{4m}R(28),\quad R(1,10)=\beta^{10m}R(15).$$

By (83), $R(26)=R(46)$, $R(28)=R(22)$, $R(11)=(-1)^fR(1,10)$.

Hence

(130) $R(17)=\beta^{2m}R(22),\quad R(11)=(-1)^f\beta^{10m}R(15).$

Jacobi (*loc. cit.*) stated a formula involving an imaginary cube root γ of unity. Thus $\gamma=\beta^4$ or β^8. For either, his formula becomes

(131) $F(\alpha)F(\beta^4\alpha)F(\beta^8\alpha)=\alpha^{-3m'}pF(\alpha^3),\quad g^{m'}\equiv 3 \pmod p.$

We employ this only for $\alpha=\beta^9$ or β^5, and eliminate p by (25) with $n=9$ or 5. By (83), $R(19)=(-1)^fR(12)$. Hence

(132) $R(15)=(-1)^fkR(33),\quad R(12)=kR(37),\quad k=\beta^{-3m'}.$

Since $p=12f+1$, 3 is a quadratic residue of p, while 2 is a quadratic residue (m even) or a non-residue (m odd), according as f is even or odd, whence

(133) m' is even, $k^2=1;\qquad \beta^{6m}=(-1)^f.$

In $p = 6f' + 1$, f' is even. By (84) with $f = f'$, (79) and (130), (83),

$$(134) \quad R(22) = R(44), \quad R(17) = (-1)^f R(44), \quad R(14) = R(44).$$

By (26), the last gives $R(18) = R(45)$. Let $R(13) = cR(15)$. Then, by (26), $R(36) = cR(45)$. By (83), $R(36) = (-1)^f R(33)$, $R(18) = (-1)^f R(13)$. Hence $R(33) = cR(13) = c^2R(15)$. Then (132) gives $(-1)^f kc^2 = 1$. By (133),

$$(135) \quad \text{either* } k = (-1)^f, \ c = \mp 1; \quad \text{or } k = -(-1)^f, \ c = \pm \beta^3; \quad R(13) = cR(15).$$

Then by (130_2), $R(13) = dR(11)$, $d = (-1)^f c\beta^{2m}$. By (26), $R(23) = dR(14)$. By (83), $R(37) = dR(17)$. By (130), (132),

$$(136) \qquad R(12) = (-1)^f ck\beta^{4m} R(22).$$

By (79) and (85),

$$(137) \quad 2R(22) = E + F(2\beta^2 - 1), \quad 2R(44) = L + 3M(2\beta^2 - 1).$$

By (75) with $d = 3$, $E = 4$, $e = 12$, we find that

$$(138) \quad (0j)_4 = (0j) + (4j) + (8j) + (8, 8 + j) + (0, 8 + j) + (4, 8 + j) + (4, 4 + j) + (8, 4 + j) + (0, 4 + j).$$

27. *Case* $e = 12$, f *odd*. By (15),

16 = 07, 17 = 05, 18 = 15, 25 = 19, 26 = 08, 27 = 15, 28 = 04, 29 = 14; 2, 10 = 24, 33 = 30, 34 = W, 35 = Z, 36 = 09, 37 = 19, 38 = 14, 39 = 03; 3, 10 = 13; 3, 11 = 23, 40 = 22, 41 = 32, 43 = 31, 44 = 20, 45 = V, 46 = X, 47 = Z, 48 = 24, 49 = 13; 4, 10 = 02; 4, 11 = 12, 50 = 11, 51 = 21, 52 = 31, 53 = 32, 54 = 21, 55 = 10, 56 = Y, 57 = V, 58 = W, 59 = 23; 5, 10 = 12; 5, 11 = 01, 60 = 00, 61 = 10, 62 = 20, 63 = 30, 64 = 22, 65 = 11, 66 = 00, 67 = 10, 68 = 20, 69 = 30; 6, 10 = 22; 6, 11 = 11, 70 = 10, 71 = Y, 72 = V, 73 = W, 74 = 23, 75 = 12, 76 = 01, 77 = 11, 78 = 21, 79 = 31; 7, 10 = 32; 7, 11 = 21, 80 = 20, 81 = V, 82 = X, 83 = Z, 84 = 24, 85 = 13, 86 = 02, 87 = 12, 88 = 22, 89 = 32; 8, 10 = 42; 8, 11 = 31, 90 = 30, 91 = W, 92 = Z, 93 = 09, 94 = 19, 95 = 14, 96 = 03, 97 = 13, 98 = 23, 99 = 30; 9, 10 = 31; 9, 11 = 32; 10, 0 = 22; 10, 1 = 23; 10, 2 = 24; 10, 3 = 19; 10, 4 = 08; 10, 5 = 15; 10, 6 = 04; 10, 7 = 14; 10, 8 = 24; 10, 9 = W; 10, 10 = 20; 10, 11 = 21; 11, 0 = 11; 11, 1 = 12; 11, 2 = 13; 11, 3 = 14; 11, 4 = 15; 11, 5 = 07; 11, 6 = 05; 11, 7 = 15; 11, 8 = 19; 11, 9 = Z; 11, 10 = V; 11, 11 = 10;

* We shall see that in some cases the ambiguity of the sign of c may be removed by choice of the primitive root g, while in the remaining case the sign is fixed by the condition that the (ij) be integers.

where $X = (0, 10)$, $Y = (0, 11)$, $Z = (1, 10)$, $V = (1, 11)$, $W = (2, 11)$. Hence the 144 numbers (ij) reduce to 31:

$$(139)\qquad \begin{gathered} 00, \cdots, 09, 10, \cdots, 15, 19, 20, \cdots, 24, 30, 31, 32, 42, \\ X, Y, Z, V, W. \end{gathered}$$

Write $[ij]$ for $(ij)_6$ for the ten $[ij]$ in

$$(140)\qquad \begin{gathered} (01) = [01] - (07) - 2(10), \quad 02 = [02] - (08) - 2(20), \\ (03) = [03] - (09) - 2(30), \quad 04 = [04] - X - 2(22), \\ (05) = [05] - Y - 2(11), \quad 06 = [00] - 3(00), \\ (12) = [12] - V - 15 - 21, \quad (13) = [13] - W - 19 - 31, \\ (14) = [14] - Z - (23) - (32), \quad (42) = [24] - 3(24), \end{gathered}$$

which follow from (77). Then (17) reduce to (87) with f replaced by $2f$, and

$$(141)\qquad \begin{aligned} (07) &= a + f + Y + W - 10 + 11 - 15 + 21 + 23 + 31 + 32, \\ (08) &= b - \tfrac{1}{2}(f+1) + X + Z - W - 00 - 10 - 11 - 15 - 19 \\ &\quad - 2(20) - 21 - 2(24) - 30 + 32, \end{aligned}$$

$$(142)\qquad (22) = \tfrac{1}{2}(f-1) - 00 - 10 - 11 - 20 - 30,$$

$$(143)\qquad a = -[05] - [12] - [13] - [14], \quad b = [02] + [12] + [13] + [24].$$

By (27), $R(33) = h + 2n\beta^3$, $p = h^2 + 4n^2$, where

$$\begin{aligned} h = &- 00 - 01 + 3(02) - 03 - 04 - 05 + 06 - 07 - 08 - 09 + 3X - Y \\ &+ 2\{10 + 11 - 12 - 13 + 14 + 15 + 19 - Z - V - 20 + 21 - 22 \\ &+ 23 - 24 + W + 30 - 31 - 32 + 42\}, \end{aligned}$$

$$\begin{aligned} n = &- 01 + 03 - 05 + 07 - 09 + 2(12) - 2(13) + 2(31) - 2(32) \\ &+ Y + 2Z - 2V. \end{aligned}$$

In $p = 12f + 1 = 4F + 1$, $F = 3f$ is odd. By § 12,

$$(144)\qquad p = x^2 + 4y^2, \ x \equiv 1 \pmod 4, \ 16(02)_4 = p + 1 - 6x, \ y = (03)_4 - (01)_4.$$

By (138) we find that $n = y$, and by (140),

$$(145)\qquad \tfrac{1}{3}(02)_4 = \tfrac{1}{3}\{[00] + 3[02] + 2[24]\} - (00) - (08) - 2(20) - 2(24) + X.$$

To h add (17) for $k = 0$ and eliminate the left members of (140) and (142). We get

$$\begin{aligned} h &\equiv -1 + 2\{[00] - [12] - [13] + [14] + [24]\} \pmod 4 \\ &\equiv -1 + 2\{[03] + [13] + [14] + 1\} \equiv -1 + 2(f+1) \equiv -1, \end{aligned}$$

by (88) and (87_4). Hence $h = -x$ and

$$(146)\qquad R(33) = -x + 2y\beta^3.$$

From the values in (27) we eliminate the left members of (140) and get

$$(147)\quad \begin{aligned} &R(15)=z+\beta^3 w,\ R(17)=\rho+\beta^2\sigma,\ R(11)=H+\beta G+\beta^2 C+\beta^3 D,\\ &R(12)=R+\beta S+\beta^2 T+\beta^3 U,\ R(13)=J+\beta K+\beta^2 P+\beta^3 Q;\end{aligned}$$

$$\begin{aligned}
z &= z'-4(00)+2(08)+4(10)+4(11)-2(15)-2(19)-2(20)\\
&\quad -2(21)-6(22)+4(24)+4(30)+2(32)-2X+2Z-2W,\\
z' &= [00]-[01]-[03]+2[04]-[05]+[12]+[13]-[14]-[24],\\
w &= w'+2\{07+2(09)+10+11-15-2(19)-21-3(23)+2(30)\\
&\quad -31-32+Y-2Z-2V+W\},\\
w' &= -[01]-2[03]-[05]+2[12]+[13]+3[14],\\
\rho &= \rho'+4[-00+10-15+19-21-2(24)-30-X+W,\\
\rho' &= [00]-[01]+[03]+[04]+2[12]-2[13]+2[24],\\
\sigma &= \sigma'+4\{08-10+11-19+2(20)-32+X-Z-W\},\\
\sigma' &= [01]-3[02]-[04]-[05]+2[13]+2[14],\\
H &= H'+4\{-00-10+15-19+21-2(24)+30-X-W\},\\
H' &= [00]+[01]-[03]+[04]-2[12]+2[13]+2[24],\\
G &= 2[05]-2[01]-4[13]+4\{07+10-11+19+2(31)+32\\
&\quad -Y-Z+W,\\
C &= C'+4\{08+10-11+19+2(20)+32+X+Z+W\},\\
C' &= -[01]-3[02]-[04]+[05]-2[13]-2[14],\\
D &= -2[03]-2[05]-4[12]\\
&\quad +4\{09+11+15+21+30-32+Y+Z+2V\},\\
R &= R'+2\{08-2(00)+2(11)+15-20+21-2(24)-2(30)+32+Z\},\\
R' &= -[00]+[03]-[05]-[12]-[14]+[24],\\
S &= S'+2\{Y-2Z-W-07-10+11+2(19)-3(23)+31-32\},\\
S' &= [01]-[05]-[13]+3[14],\\
T &= T'+2\{-08+2(10)-2(11)-19+20-3(22)-32-X-Z-W\},\\
T' &= -[01]+2[04]+[05]+[13]+[14],\\
U &= U'+2\{07-2(09)+10+15-2(19)+21-2(30)-31+2V+W\},\\
U' &= -[01]+2[03]-2[12]+[13].\\
J &= J'+2\{-2(00)-08+2(11)+15+20\\
&\quad +21+2(24)-2(30)+32+Z\},\\
K &= K'+2\{07+10-11-2(15)+2(21)-23-31-32-Y-W\},\\
P &= P'+2\{08+2(10)-2(11)-19-20-3(22)-32+X-Z-W\},\\
Q &= Q'+2\{2Y-W+07+10+2(11)+15-21+2(23)-31+2(32)\},\\
J' &= [00]+[03]-[05]-[12]-[14]-[24],\\
K' &= [05]-[01]+[13]+[14],\\
P' &= -[01]-2[04]+[05]+[13]+[14],\\
Q' &= -[01]-2[05]+[13]-2[14].
\end{aligned}$$

The ten $[ij]$ were found in § 18 and are here regarded as known. The 31 numbers (139) are connected by the $10+3$ equations (140)-(142), the 16 whose left members are $z, \cdots, Q$, and the two final equations (144), amplified by (145) and (138). These 31 linear equations uniquely determine the 31 numbers (139) and hence all 144 of the (ij).

We seek especially (00) and hence (06) by (140). We shall find 00, 24 and 30 simultaneously by three linear equations:

$$(148)\quad \begin{aligned} &2\rho+\sigma+2P+4J=s+2P'+4J'+6f-6-36(00)-36(30),\\ &2\rho+\sigma+4R+2T-2z=s+4R'+2T'-2z'+4b-2f-2\\ &\qquad\qquad\qquad +12(00)-48(24)-36(30),\\ &2\rho+\sigma+2H+C+8/3(02)_4=s+2H'+C'+8/3\{[00]+3[02]\\ &\qquad\qquad\qquad +2[24]\}-24(00)-48(24), \end{aligned}$$

$$\begin{aligned} s=2[00]-[01]-3[02]+2[03]+[04]\\ -[05]+4[12]-2[13]+2[14]+4[24]. \end{aligned}$$

I. Let 2 be a cubic residue of p. Then m is an odd multiple of 3, and $\beta^{2m}=-1$.

In (148), insert the values (91) of the $[ij]$, and solve. We get

$$(149)\quad \begin{aligned} &144(00)=p-23-20A+2x-2\tau+2\phi, \quad \tau=2\rho+\sigma+2P+4J,\\ &144(30)=p-11+4A-2x-2\tau-2\phi, \quad \phi=4R+2T-2z-2H-C,\\ &144(24)=p+1-2A+2x+\tau-\phi-3(2\rho+\sigma)-6H-3C. \end{aligned}$$

By I of § 18, $L=E=-2A$, $F=2B=3M$. By (137), $R(22)=R(44)=-A-B+2\beta^2B$. By (130), $2\rho+\sigma=2A$, $G=C=0$, $H=z$, $D=w$.

I_1. Let 3 be a biquadratic residue of p. Then $k=+1$ in (132). By also (135), (136),

$$\begin{gathered} R(15)=-R(33), \quad R(13)=\pm\beta^3R(15), \quad R(12)=\mp\beta^3R(22),\\ z=x, \quad w=-2y, \quad K=P=0, \quad J=\pm 2y, \quad Q=\pm x, \quad R=T=0,\\ S=\pm 2B, \quad U=\pm(A-B). \end{gathered}$$

The upper signs are replaced by the lower when g is replaced by g^r, $r\equiv-1 \pmod{12}$. By Theorem 10 we see that x, z, A, E, L, K, G, S, 00, 30, 24 are unaltered; $y, w, B, F, M, P, C, T, \sigma$ are changed in sign; and, if J becomes J_1, then $J_1=J+P$, $Q_1=-Q-K$, $H_1=H+C$, $D_1=-D-G$, $R_1=R+T$, $U_1=-U-S$, $\rho_1=\rho+\sigma$. We get

$$(150)\quad \begin{aligned} &144(00)=p-23-24A-6x\mp 16y, \quad 144(30)=p-11+6x\mp 16y,\\ &144(24)=p+1-6A\pm 8y. \end{aligned}$$

I_2. Let 3 be a biquadratic non-residue of p. Then $k=-1$, $z=-x$, $w=2y$, $K=P=S=U=0$, $J=\pm x$, $Q=\mp 2y$, $R=\pm(A+B)$, $T=\mp 2B$,

$$
\begin{aligned}
&144(00)=p-23-24A+10x\pm 8(A-x),\\
(151)\qquad &144(30)=p-11-10x\mp 8(A+x),\\
&144(24)=p+1-6A+4x\mp 4(A-x).
\end{aligned}
$$

The signs are not affected by the choice of the primitive root g, but are determined by the fact that the right members shall be integers. The upper signs hold if $p=157$, the lower if $p=397$ or 997 (the only p's < 1000).

For $m\equiv 1$ or $4 \pmod 6$, $E,\cdots,M$, (ij) are given by III of § 18. Then (148) give

$$
\begin{aligned}
&144(00)=p-23+2x-8A+6B+2(v-u),\\
(152)\quad &144(30)=p-11-2x+4A-6B-2(v+u),\\
&144(24)=p+1+2x+10A+6B-v+u-3(2\rho+\sigma+2H+C),
\end{aligned}
$$

$$u=2\rho+\sigma+2P+4J,\quad v=4R+2T-2z-2H-C.$$

II_1. Let $2m\equiv 2 \pmod{12}$, $k=-1$. Then (130)-(137) give $C=z=-x$, $w=2y$, $H=x$, $G=-2y$, $D=0$, $2\rho+\sigma=-A+3B$, $J=\pm x$, $Q=\mp 2y$, $R=\pm(A+B)$, $T=\mp 2B$, $K=P=S=U=0$.

$$
\begin{aligned}
&144(00)=p-23+4x-6A\pm 8(A-x),\\
(153)\qquad &144(24)=p+1-2x+12A\mp 4(A-x),\\
&144(30)=p-11-4x+6A-12B\mp 8(A+x).
\end{aligned}
$$

II_2. Let $2m\equiv 2 \pmod{12}$, $k=1$. Then

$z=C=x$, $H=-x$, $Q=\pm x$, $G=2y$, $w=-2y$, $J=\pm 2y$,
$D=K=P=R=T=0$, $S=\pm 2B$, $U=\pm(A-B)$, $2\rho+\sigma=-A+3B$,

$$
(154)\quad
\begin{aligned}
&144(00)=p-23-6A\mp 16y,\quad 144(24)=p+1+6x+12A\pm 8y,\\
&144(30)=p-11+6A-12B\mp 16y.
\end{aligned}
$$

We may take the upper signs. For, if g be replaced by a new primitive root g^r, $r\equiv 7 \pmod{12}$, k is unchanged, $2m$ is unaltered modulo 12, y is changed in sign, while

$$(155)\qquad (00),\ (30),\ (24),\ x,\ A$$

and B are unaltered.

If $r\equiv -1 \pmod{12}$, we saw under I_1 that (155) are unaltered while

y and B are changed in sign. If $r \equiv 5 \pmod{12}$, (155) and y are unaltered, while B is changed in sign. This proves

III$_1$. If $2m \equiv 10 \pmod{12}$, $k = -1$, change the sign of B in II$_1$.

III$_2$. If $2m \equiv 10 \pmod{12}$, $k = 1$, change the sign of B in II$_2$. The upper signs hold when g is chosen properly. There are no further cases with f odd, since m is then odd.

28. *Case $e = 12$, f even.* We replace (144), (145) by

$$p = x^2 + 4y^2,\quad x \equiv 1 \pmod 4,\quad 16(00)_4 = p - 11 - 6x,\quad y = (01)_4 - (03)_4,$$
$$\tfrac{1}{3}(00)_4 = \tfrac{1}{3}\{[00] + 3[02] + 2[24]\}$$
$$- (02) + (04) - (06) - 2(26) - 2(2,10).$$

The further formulas for f odd hold here if we change the signs of the expressions for R, S, T, U (and hence of $R', \cdots$), replace (k, h) by $(k, h + 6)$, and change the constant terms as follows: to $-\frac{1}{2}f$ in (141$_2$), to $\frac{1}{2}f$ in (142), suppress -6 and -2 from the first and second equations (148), replace $(02)_4$ by $(00)_4$ in the third.

I. Let 2 be a cubic residue of p. Then $\beta^{2m} = +1$. Now $2\rho + \sigma = -2A$. Change the constant terms in (149) to $-11, 1, 1$.

I$_1$. Let 3 be a biquadratic residue of p. Then $k = 1$,

$$H = z = -x,\quad D = w = 2y,\quad J = \pm x,\quad Q = \mp 2y,\quad R = \pm (A + B),$$
$$T = \mp 2B,\quad G = C = K = P = S = U = 0,$$

$$\begin{aligned} 144(06) &= p - 11 + 10x - 16A \mp 8(A + x), \\ 144(36) &= p + 1 - 10x + 8A \pm 8(A - x), \\ 144(2,10) &= p + 1 + 4x + 2A \pm 4(A + x). \end{aligned} \tag{156}$$

I$_2$. Let 3 be a biquadratic non-residue of p. Then $k = -1$,

$$H = z = x,\ D = w = -2y,\ J = \pm 2y,\ Q = \pm x,\ S = \pm 2B,\ U = \pm (A - B),$$
$$G = C = K = P = R = T = 0,$$

$$\begin{aligned} &144(06) = p - 11 - 6x - 16A \mp 16y,\quad 144(2,10) = p + 1 + 2A \pm 8y, \\ &144(36) = p + 1 + 6x + 8A \mp 16y. \end{aligned} \tag{157}$$

For $m \equiv 4 \pmod 6$, the products of (06), (36), (2, 10) by 144 are given by (152) with the constant terms replaced by $-11, 1, 1$ and

II$_1$. Let $2m \equiv 8 \pmod{12}$, $k = -1$. Then

$$C=z=x,\ w=-2y,\ G=2y,\ H=-x,\ J=\pm 2y,\ 2\rho+\sigma=A-3B,$$
$$Q=\pm x,\ S=\pm 2B,\ U=\pm(A-B),\ D=K=P=R=T=0,$$

$$(158)\quad \begin{array}{l} 144(06)=p-11-10A+12B\mp 16y,\ 144(36)=p+1+2A\mp 16y, \\ 144(2,10)=p+1+6x+8A+12B\pm 8y. \end{array}$$

The sign of y is changed when g is replaced by g^r, $r\equiv 7 \pmod{12}$.

II_2. Let $2m\equiv 8 \pmod{12}$, $k=+1$. Then

$$C=z=-x,\ w=2y,\ H=x,\ G=-2y,\ J=\pm x,\ 2\rho+\sigma=A-3B,$$
$$Q=\mp 2y,\ R=\pm(A+B),\ T=\mp 2B,\ D=K=P=S=U=0,$$

$$(159)\quad \begin{aligned} 144(06)&=p-11+4x-10A+12B\mp 8(A+x), \\ 144(36)&=p+1-4x+2A\pm 8(A-x), \\ 144(2,10)&=p+1-2x+8A+12B\pm 4(A+x). \end{aligned}$$

III. Let $2m\equiv 4 \pmod{12}$. When g is replaced by g^r, $r\equiv -1 \pmod{12}$, (06), (36), $(2,10)$, A, x remain unaltered, while B and y are changed in sign. Making the latter change in II_1 and II_2, we obtain the present values of $144(06)$, etc.

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SPINORS IN n DIMENSIONS.

By Richard Brauer and Hermann Weyl.

Introduction and Summary. Let $\mathfrak{d}_n$ be the group of orthogonal transformations o:

$$(1) \qquad x_i \to \sum_{k=1}^{n} o(ik)x_k \qquad (i=1,2,\cdots,n)$$

of the n-dimensional space, and $\mathfrak{d}_n^+$ the subgroup of proper transformations, having determinant $+1$ and not -1. We shall first operate within the continuum of all complex numbers, whereas the particular conditions prevailing under restriction to real variables will be studied only at the end of the paper (§§ 8 and 10). A given representation $\Gamma : o \to G(o)$ of degree N defines a certain kind of "covariant quantities": a quantity characterized by N numbers $a_1, \cdots, a_N$ relative to an arbitrary Cartesian coördinate system in the underlying n-dimensional Euclidean space will be called *a quantity of kind* Γ, provided the components a_K experience the linear transformation $G(o)$ under the influence of the coördinate transformation o. The quantity is called *primitive* if the representation is irreducible. The proposition that every representation breaks up into irreducible parts, states that the most general kind of quantities is obtained by juxtaposition of several independent primitive quantities.

By a *tensor of rank* f we shall mean here what usually is called a skew-symmetric tensor: a skew-symmetric function $\alpha(i_1 \cdots i_f)$ of f indices ranging independently from 1 to n which transforms according to the law

$$\alpha(i_1 \cdots i_f) \to \sum_{k_1,\ldots,k_f=1}^{n} o(i_1k_1) \cdots o(i_fk_f) \cdot \alpha(k_1 \cdots k_f)$$

under the influence of the rotation o. The tensors of rank f form the substratum of a representation Γ_f of degree $\binom{n}{f}$.

We often have to distinguish between even and odd dimensionality, and we shall accordingly put $n=2\nu$ or $n=2\nu+1$. Let us use the notation $\nu=\langle n\rangle$ and in passing notice the congruence

$$\tfrac{1}{2}n(n-1) \equiv \langle n\rangle \pmod 2.$$

E. Cartan developed a general method of constructing irreducible representations of $\mathfrak{d}_n$ (or any other semi-simple group) by considering the in-

finitesimal operations, and he found † as the building stones of the whole edifice the tensor representations Γ_f together with *one further double-valued representation* $\Delta : o \rightarrow S(o)$ *of degree* 2^ν. The quantities of kind Δ are called *spinors*. In the four-dimensional world this kind of quantities has come to its due honors by Dirac's theory of the spinning electron. Cartan, according to his standpoint, states the transformation law $S(o)$ of spinors only for the infinitesimal rotations o. Here we shall give a simple finite description of the representation Δ and shall derive from it by the simplest algebraic means the main properties of the spinors. One will be able to judge by this theory to what extent recent investigations about spinor calculus reveal those essential features that stay unchanged for higher dimensions. One of the chief results will be that Dirac's equations of the motion of an electron and the expression for the electric current are uniquely determined even in the case of arbitrary dimensionality.

Our investigation will be arranged as follows: we start (§ 2) with a certain associative algebra Π of order $2^{2\nu}$ which proves to be a complete matrix algebra in 2^ν dimensions, and leads to the desired definition of Δ (§ 3). We shall first get Δ as a collineation representation such that only the ratios of the spinor components have a meaning. In the case of even dimensionality $n = 2\nu$ we shall prove (§ 3) that the product $\Delta \times \check{\Delta}$ of Δ by the contragredient representation $\check{\Delta}$ splits up according to the equivalence:

$$\Delta \times \check{\Delta} \sim \Gamma_0 + \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n,$$

whereas in the odd case

$$\Delta \times \check{\Delta} \sim \Gamma_0 + \Gamma_2 + \Gamma_4 + \cdots + \Gamma_{n-1}$$

(§ 5). The collineation representation Δ can be normalized so as to give an ordinary, though double-valued representation Δ satisfying the equivalence $\check{\Delta} \sim \Delta$ (§§ 4, 5). If one restricts oneself to the proper orthogonal transformations in a space of even dimensionality, Δ splits up into two representations Δ^+ and Δ^- each of degree $2^{\nu-1}$ (§ 6). The four products of the type $\Delta \times \check{\Delta}$ will be determined individually for $\Delta = \Delta^+$ or Δ^-, and so will the equivalences of type $\check{\Delta} \sim \Delta$. The transition from our finite to Cartan's infinitesimal description can be easily performed (§ 7). In considering real transformations only, the differences of the inertial index have to be taken into account (§ 8); it will be proved that $\check{\Delta}$ is equivalent to Δ again—but for a sign the determination of which is of peculiar interest and closely related

† *Bulletin Société Mathématique de France*, vol. 41 (1913), p. 53. Compare also Weyl, *Mathematische Zeitschrift*, vol. 24 (1926), p. 342.

to the inertial index. Irreducibility and equivalence of the occurring representations will be ascertained in § 9, and the relation to physics will be discussed in § 10. In parts of the investigation we must have recourse to the law of duality of tensors and tensor representations Γ_f as formulated in the preliminary § 1. The last section (§ 11) is devoted to the demonstration of a well-known fundamental proposition concerning the automorphisms of the complete matrix algebra, a proposition indispensable for the definition of Δ.

1. *Duality of tensors.* Γ_n is the representation of degree 1 of the full rotation group $\mathfrak{d}_n$ associating the signature $\sigma(o)$ with the rotation $o: \sigma(o) = +1$ for the proper, $\sigma(o) = -1$ for the improper rotations. Any representation $\Gamma: o \to G(o)$ gives rise to another representation $\sigma\Gamma: o \to \sigma(o)G(o)$, coinciding with Γ under restriction to $\mathfrak{d}_n^+$.

The equation

$$\alpha^*(i'_1 \cdots i'_{n-f}) = \alpha(i_1 \cdots i_f) \tag{2}$$

in which $i_1 \cdots i_f\, i'_1 \cdots i'_{n-f}$ denotes any even permutation of the figures from 1 to n, associates a tensor α^* of rank $n - f$ with every tensor α of rank f. This relation is invariant with respect to proper orthogonal transformations. Thus the *law of duality* $\Gamma_{n-f} \sim \Gamma_f$ prevails for the tensor representations Γ_f of $\mathfrak{d}_n^+$. When taking the improper orthogonal transformations into consideration it is to be replaced by

$$\Gamma_{n-f} \sim \sigma\Gamma_f.$$

In the case of an even number of dimensions $n = 2\nu$, the representation Γ_ν deserves particular attention. It satisfies the equivalence $\sigma\Gamma_\nu \sim \Gamma_\nu$. (2) or rather

$$\alpha^*(i'_1 \cdots i'_\nu) = i^\nu \cdot \alpha(i_1 \cdots i_\nu) \tag{3}$$

now establishes a transformation $\alpha \to \alpha^*$ of the space of the tensors of rank ν *upon itself.* We added the factor i^ν in order to make this transformation involutorial: $\alpha^{**} = \alpha$; for if $i_1 \cdots i_\nu\, i'_1 \cdots i'_\nu$ is an even permutation, $i'_1 \cdots i'_\nu\, i_1 \cdots i_\nu$ has the character $(-1)^\nu$. We may distinguish between positive and negative tensors of rank ν according as $\alpha^* = \alpha$ or $\alpha^* = -\alpha$. Any tensor of rank ν can be decomposed in a unique manner into a positive and a negative part:

$$\alpha = \tfrac{1}{2}(\alpha + \alpha^*) + \tfrac{1}{2}(\alpha - \alpha^*).$$

Hence, as a representation of the group $\mathfrak{d}^+_{2\nu}$, Γ_ν splits up into two representations $\Gamma_\nu^+ + \Gamma_\nu^-$ of half the degree.

2. *The algebra* Π. Our procedure is exactly the same as followed by

Dirac in his classical paper on the spinning electron.† We introduce n quantities p_i which turn the fundamental quadratic form into the square of a linear form:

$$(4) \qquad x_1^2 + \cdots + x_n^2 = (p_1 x_1 + \cdots + p_n x_n)^2.$$

For this purpose we must have

$$(5) \qquad p_i^2 = 1, \qquad p_k p_i = -p_i p_k \qquad (k \neq i).$$

The quantities p_i engender an algebra consisting of all linear combinations of the 2^n units

$$(6) \qquad e_{\alpha_1 \ldots \alpha_n} = p_1^{\alpha_1} \cdots p_n^{\alpha_n} \qquad (\alpha_1, \cdots, \alpha_n \text{ integers mod } 2).$$

The recipe for multiplication of the units reads, according to (5):

$$e_{\alpha_1 \ldots \alpha_n} \cdot e_{\beta_1 \ldots \beta_n} = (-1)^\delta \cdot e_{\gamma_1 \ldots \gamma_n}; \qquad \gamma_i = \alpha_i + \beta_i, \quad \delta = \sum_{i>k} \alpha_i \beta_k.$$

One easily convinces oneself that this rule of multiplication is associative.

One may write the most general quantity a of our algebra in the form

$$(7) \quad a = \cdots + (1/f!) \sum_{(i_1, \ldots, i_f)} \alpha(i_1 \cdots i_f) p_{i_1} \cdots p_{i_f} + \cdots \quad (f = 0, 1, \cdots, n),$$

splitting a into parts according to the number f of the different factors p. Since the product of f different p's like $p_{i_1} \cdots p_{i_f}$ is skew-symmetric with respect to the indices $i_1 \cdots i_f$, one will choose the coefficients $\alpha(i_1 \cdots i_f)$ in (7) also skew-symmetric; one is then allowed to extend the sum Σ in (7) over the indices $i_1, \cdots, i_f$ independently from 1 to n. Consequently the quantity a is equivalent to a "tensor set" consisting of $n + 1$ tensors, one of each of the ranks $0, 1, \cdots, f, \cdots, n$. The addition of two tensor sets and the multiplication of a set by a number has the trivial significance within the algebra Π. But how are we to express the multiplication of two tensor sets a and b? It suffices to describe the case of an a containing merely one tensor α of rank f, and a b containing merely one tensor β of rank g (whereas the other parts vanish). The product splits into different parts according to the number r of coincidences among the indices of α and β. As

$$p_{i_1} \cdots p_{i_{f-r}} \, p_{l_1} \cdots p_{l_r} \cdot p_{l_1} \cdots p_{l_r} p_{k_1} \cdots p_{k_{g-r}} = (-1)^{r(r-1)/2} \cdot p_{i_1} \cdots p_{i_{f-r}} p_{k_1} \cdots p_{k_{g-r}},$$

one gets as part r of the product essentially the "contraction"

† *Proceedings of the Royal Society* (A), vol. 117 (1927), p. 610; vol. 118 (1928), p. 351.

$$(8)\quad \gamma(i_1\cdots i_{f-r}\,k_1\cdots k_{g-r}) = \sum_{(l_1,\ldots,l_r)} \alpha(i_1\cdots i_{f-r}\,l_1\cdots l_r)\cdot\beta(l_1\cdots l_r\,k_1\cdots k_{g-r}).$$

This process, however, has to be followed by "alternation," i. e. alternating summation over all permutations of the $f+g-2r$ indices in γ. Since γ is already skew-symmetric with respect to the $f-r$ indices i and the $g-r$ indices k, it is sufficient to extend an alternating sum over all "mixtures" of the indices $i_1\cdots i_{f-r}$ with the indices $k_1\cdots k_{g-r}$. This will be indicated by the symbol $\boldsymbol{M}$. By taking into consideration the factor $1/f!$ attached to the f-th term in (7) and the several distributions of the r equal indices $l_1\cdots l_r$ among the indices of α and β, one gets finally the result: The "product" of the two tensors α and β is a tensor set in which only tensors of rank $f+g-2r$ appear; the integer r is limited by the bounds

$$r \geqq 0, \qquad 2r \geqq f+g-n, \qquad r \leqq f, \quad r \leqq g.$$

The part r is given by

$$(-1)^{\langle r\rangle}(1/r!)\cdot\boldsymbol{M}\gamma(i_1\cdots i_{f-r}\,k_1\cdots k_{g-r})$$

where γ denotes the contraction (8).—We are not so much interested in the exact description of this process of multiplication as in the fact *that it is orthogonally invariant.*

3. *Spinors in a space of even dimensionality.* In this section we suppose $n=2\nu$ to be even. The algebra Π is known to the quantum theorist from the process of "superquantizing" that allows the passage from the theory of a single particle to the theory of an undetermined number of equal particles subjected to the Fermi statistics. This connection at once yields a definite representation $p_i \to P_i$ by matrices P_i of order 2^ν. Into its description enter the two-rowed matrices

$$\mathbf{1} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad \mathbf{1}' = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}, \quad P = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \quad Q = \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix}.$$

The two rows and columns will be distinguished from each other by the signs $+$ and $-$. $\mathbf{1}'$, P, Q anticommute with each other; their squares are $=\mathbf{1}$. Besides $p_1,\cdots,p_{2\nu}$ we sometimes use the notation $p_1,\cdots,p_\nu$, $q_1,\cdots,q_\nu$. The representation then is given by

$$(9)\quad \begin{aligned} p_\alpha \to P_\alpha &= \mathbf{1}'\times\cdots\times\mathbf{1}'\times P\times\mathbf{1}\times\cdots\times\mathbf{1}, \\ q_\alpha \to Q_\alpha &= \mathbf{1}'\times\cdots\times\mathbf{1}'\times Q\times\mathbf{1}\times\cdots\times\mathbf{1}. \end{aligned} \qquad (\alpha=1,\cdots,\nu).$$

On the right side we have ν factors; the factors P, Q respectively, occur at the α-th place. The rows and columns of our matrices or the coördinates x_A in

the 2^ν-dimensional representation space, according to the notation introduced, are distinguished from each other by a combination of signs $(\sigma_1, \sigma_2, \cdots, \sigma_\nu)$, $(\sigma_\alpha = \pm)$. One verifies at once that the desired rules prevail:

$$P_i^2 = 1, \qquad P_k P_i = - P_i P_k \qquad (i \neq k). \tag{10}$$

In this manner we have established a definite *representation* $x \to X$ of degree 2^ν for the algebra Π. We maintain that *all matrices X appear here as images of elements x of the algebra.* As the algebra Π is of the same order $2^{2\nu} = (2^\nu)^2$ as the algebra consisting of all matrices in the 2^ν-dimensional space, the relation $x \rightleftarrows X$ is a one-to-one isomorphic mapping of Π upon the complete matrix algebra of the 2^ν-dimensional "spin space": the algebra Π is isomorphic to the complete matrix algebra in spin space. In order to prove our statement, let us compute the matrix U_α representing $u_\alpha = ip_\alpha q_\alpha$:

$$U_\alpha = iP_\alpha Q_\alpha = \mathbf{1} \times \cdots \times \mathbf{1} \times \mathbf{1}' \times \mathbf{1} \times \cdots \times \mathbf{1} \tag{11}$$

and then

$$U_1 \cdots U_{\alpha-1} P_\alpha = \mathbf{1} \times \cdots \times \mathbf{1} \times P \times \mathbf{1} \times \cdots \times \mathbf{1} \tag{11'}$$

together with $U_1 \cdots U_{\alpha-1} Q_\alpha$. (The factors different from $\mathbf{1}$ occur at the α-th place.) Thus the following elements

$$\tfrac{1}{2}(1 + u_\alpha) = z_\alpha^{++}, \qquad \tfrac{1}{2}u_1 \cdots u_{\alpha-1}(p_\alpha - iq_\alpha) = z_\alpha^{+-},$$
$$\tfrac{1}{2}u_1 \cdots u_{\alpha-1}(p_\alpha + iq_\alpha) = z_\alpha^{-+}, \qquad \tfrac{1}{2}(1 - u_\alpha) = z_\alpha^{--}$$

are represented by products similar to (11) but containing one of the matrices

$$\begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}$$

at the α-th place. Consequently the image of the element $\prod_{\alpha=1}^{\nu} (z_\alpha^{\sigma_\alpha \tau_\alpha})$ is the matrix containing a term different from 0, namely 1, only at the crossing point of the row $\sigma_1 \cdots \sigma_\nu$ with the column $\tau_1 \cdots \tau_\nu$ $(\sigma_\alpha = \pm, \tau_\alpha = \pm)$.

We are now in a position to establish the connection with the rotations $o = \| o(ik) \|$ in the n-dimensional space (Method A). We change, by means of the orthogonal matrix $o(ik)$

$$P_i \to P^*_i = \sum_{k=1}^{n} o(ki) P_k, \qquad P_i = \sum_{k=1}^{n} o(ik) P_k^* \tag{12}$$

and we observe at once that the new P^*_i, like the old ones, satisfy the relations (10). Consequently $p_i \to P^*_i$ defines a new representation of our algebra Π. *Since the full matrix algebra, however, allows only inner automorphisms,*†

† See the proof in § 11.

this representation has to be equivalent to the original one; that is, there exists a non-singular matrix $S(o)$ such that

$$(13) \qquad P^*_i = S(o) P_i\, S(o)^{-1} \qquad (i = 1, 2, \cdots, n).$$

$S(o)$ is determined by this equation but for a numerical factor, the "gauge factor": $S(o)$ is to be interpreted in the "homogeneous" sense, not as an affine transformation of the 2^ν-dimensional vector space, but as a collineation of the projective space consisting of its rays. After fixing the gauge factors for two rotations o, o' and their product $o'o$ in an arbitrary manner, we necessarily have a relation like

$$(14) \qquad S(o'o) = c \cdot S(o') S(o).$$

Consequently we are dealing with a *collineation representation* of degree 2^ν of the rotation group, the so-called *spin representation* $\Delta : o \rightarrow S(o)$.

The same connection can be described as follows (Method B). Orthogonal transformation of the tensors of an arbitrary tensor set defines an automorphic mapping $x \rightarrow x^*$ of the algebra Π of the tensor sets upon itself. Such a mapping however, in the representation $x \rightarrow X$ of the tensor sets by matrices X of order 2^ν, is necessarily displayed in the form

$$X \rightarrow X^* = SXS^{-1} \quad (S \text{ independent of } x).$$

Let us write down this equation in components: $X = \| x_{JK} \|$; it then reads

$$x^*_{JK} = \sum_{R,T} s_{JR}\, \check{s}_{KT}\, x_{RT}.$$

$\check{S} = \| \check{s}_{JK} \|$ is the matrix contragredient to S. Hence the components x_{JK} experience the transformation $S \times \check{S}$ and this proves the reduction

$$(15) \quad \Delta \times \check{\Delta} \sim \Gamma_0 + \Gamma_1 + \cdots + \Gamma_n \sim \left\{ \begin{array}{l} \Gamma_0 + \ \Gamma_1 + \cdots + \Gamma_{\nu-1} + \\ \sigma\Gamma_0 + \sigma\Gamma_1 + \cdots + \sigma\Gamma_{\nu-1} \end{array} \right\} + (\Gamma_\nu \sim \sigma\Gamma_\nu).$$

The quantities $\{\psi^A\}$ and $\{\phi_A\}$ of the kind Δ, $\check{\Delta}$ shall be called *covariant and contravariant spinors* respectively. Let us write the components ψ^A of a covariant spinor as a column and the components ϕ_A of a contravariant spinor as a row. Our last equation tells us that one is able to form by linear combination of the $(2^\nu)^2$ products $\phi_A\psi^B$: *one* scalar, *one* vector, *one* tensor of rank 2, etc. The scalar is, of course,

$$\phi\psi = \sum_A \phi_A \psi^A.$$

The vector has the components $\phi P_i \psi$. Indeed, in carrying out the transformation $\psi^* = S\psi$, $\phi^* = \phi S^{-1}$, one gets,

$$\phi^* P_i \psi^* = \phi S^{-1} P_i S \psi = \sum_{k=1}^{n} o(ik) \phi S^{-1} P^*_k S \psi = \sum_{k=1}^{n} o(ik) \phi P_k \psi.$$

The tensor of rank 2 has the components $\phi(P_i P_k)\psi$ $[i \neq k]$; etc. In this manner we are able to carry out the reduction (15) explicitly.

4. *Connection between covariant and contravariant spinors.* Let n be *even* as before. We propose to show that *the representation $\check{\Delta}$ is equivalent to the representation Δ.* For this purpose we observe that the relations (10) characteristic for the matrices P_i hold at the same time for the transposed matrices P'_i. According to the proposition on the automorphisms of our matrix algebra Π we already have had occasion to use, there must exist a definite non-singular matrix C such that

$$P'_i = C P_i C^{-1} \tag{16}$$

for all i. It is easy to write down C explicitly. For we have

$$P'_\alpha = P_\alpha, \qquad Q'_\alpha = -Q_\alpha \qquad (\alpha = 1, \cdots, \nu).$$

But the product $p_1 \cdots p_\nu$ commutes with the p_α and anticommutes with the q_α, if ν is odd; if ν is even the situation is reversed. Hence one can take

$$c = p_1 \cdots p_\nu \quad \text{or} \quad = q_1 \cdots q_\nu$$

according as ν is odd or even. In this way one finds in both cases:

$$C = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \times \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix} \times \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \times \cdots \qquad (\nu \text{ factors}) \tag{17}$$

and one verifies at once the relations (16).

Along with (12) we have

$$P'_i \rightarrow P^{*\prime}_i = \sum_k o(ki) P'_k.$$

This transition is expressed on the one hand in the form

$$P'_i \rightarrow S'(o)^{-1} P'_i S'(o) = \check{S}(o) P'_i \check{S}(o)^{-1}.$$

On the other hand the transformation of $P'_i = C P_i C^{-1}$ is obviously performed by means of $C S(o) C^{-1}$. Hence an equation like

$$C S(o) C^{-1} = \rho(o) \cdot \check{S}(o)$$

must hold where $\rho(o)$ is a numerical factor dependent on o. On multiplication of $S(o)$ by λ, $\check{S}(o)$ is multiplied by $1/\lambda$ and ρ is thus changed into $\rho\lambda^2$.

Hence we may dispose of the arbitrary gauge factor in S in such a way that ρ becomes $= 1$:

(18) $$\check{S}(o) = CS(o)C^{-1}.$$

This has the effect that

(19) $$(\det S)^2 = 1.$$

$S(o)$ is now uniquely determined *but for the sign.* After normalizing this sign for two rotations o, o' and the compound $o'o$ in an arbitrary manner, the composition factor c in (14) becomes $= \pm 1$; for the matrices $X = S(o'o)$ and $X = S(o')S(o)$ both satisfy the normalizing condition

$$\check{X} = CXC^{-1}.$$

Δ now is an ordinary, though double-valued representation instead of a collineation representation.

Equation (18) gives the explicit relation between the covariant and contravariant spinors: if C is the matrix $\| c_{AB} \|$ the substitution

$$\phi_A = \sum_B c_{AB}\, \psi^B$$

changes the covariant spinor ψ into a contravariant spinor ϕ.

The "square" of the double-valued representation Δ is single-valued and is decomposed, according to formula

$$\Delta \times \Delta \sim \Gamma_0 + \Gamma_1 + \cdots + \Gamma_{n-1} + \Gamma_n$$

into the tensor representations Γ_f.

5. *Odd number of dimensions.* $n = 2\nu + 1$. To our quantities $p_1, \cdots, p_{2\nu}$ a further one $p_{2\nu+1}$ has to be added, $p^2{}_{2\nu+1} = 1$, which anticommutes with the previous p_i. The representation $p_i \rightarrow P_i$ $(i = 1, \cdots, 2\nu)$ can be extended by establishing the correspondence

$$p_n \rightarrow P_n = \mathbf{1}' \times \mathbf{1}' \times \cdots \times \mathbf{1}' \qquad (n = 2\nu + 1).$$

Let ι be $= 1$ or i according as ν is even or odd. The product

(20) $$u = \iota p_1 p_2 \cdots p_n$$

commutes with all quantities of the algebra and satisfies the equation $u^2 = 1$. In the representation just described u is represented by the matrix 1. There exists a second representation of the algebra:

(21) $$p_i \rightarrow - P_i \qquad (i = 1, 2, \cdots, n)$$

in which $u \rightarrow -1$ and which thus proves to be inequivalent to the first one.

The order $2 \cdot (2^\nu)^2$ of the algebra Π this time is twice as large as the order of the algebra of all matrices X in the 2^ν-dimensional spin-space. Our isomorphic mapping $x \rightarrow X$ therefore becomes a one-to-one correspondence only after reducing Π modulo $(1 - u)$; this is accomplished by adding the condition $u = 1$ to the defining equations (5). This new algebra may be realized as a subalgebra in Π in different manners; for instance, as the algebra of the quantities x satisfying the condition $x = ux$. It is more convenient to consider the *even* quantities in Π. Their basis consists of the products of an even number of p; in (6) one has to add the restriction $\alpha_1 + \cdots + \alpha_n \equiv 0 \pmod 2$; the corresponding tensor sets contain tensors of even rank only. Any odd quantity may be written in the form ux where x is even. The arbitrary quantity $x + ux'$ of the algebra Π (x and x' even) is represented by the same matrix as the even quantity $x + x'$. Hence the correspondence $x \rightarrow X$ is a one-to-one correspondence within the algebra Π_e of the even quantities. The second representation (21) coincides with the first for the even quantities.

The procedure is now as above (Method A). Let $\| o(ik) \|$ be a proper orthogonal transformation. Then (12) yields a new representation of Π. By multiplication we get

$$U^* = \iota P^*_1 \cdots P^*_n = \det [o(ik)] \cdot U = U.$$

Hence this representation like the original one associates the matrix $+1$ (and not -1) with u; by means of $P_i \rightarrow P^*_i$ we thus map the algebra Π reduced modulo $(1 - u)$ isomorphically upon itself, and consequently an equation like

$$P^*_i = SP_iS^{-1}$$

holds. The representation $\Delta : o \rightarrow S(o)$ may be extended to the improper rotations by making the matrix $+1$ or -1 correspond to the reflection $x_i \rightarrow - x_i$ that commutes with all rotations. (Whether one chooses $+1$ or -1 does not make any difference here since the representation Δ is double-valued.)

(Method B). The orthogonal transformation o is an isomorphic mapping of the manifold of all even tensor sets upon itself. After representing this manifold by the algebra of all matrices X in 2^ν dimensions in the manner described above, o appears as an automorphism $X \rightarrow X^*$ of the complete matrix algebra: $X^* = SXS^{-1}$. One gets $S(o)$ here at the same time for all proper and improper rotations o. Furthermore, we obtain the decomposition

$$(22) \qquad \Delta \times \check{\Delta} \sim \Gamma_0 + \Gamma_2 + \cdots + \Gamma_{2\nu} \sim \Gamma_0 + \sigma\Gamma_1 + \Gamma_2 + \sigma\Gamma_3 + \cdots,$$

the last sum concluding with the term Γ_ν or $\sigma\Gamma_\nu$. Consequently there is contained in $\Delta \times \check{\Delta}$ a proper scalar, an improper vector, a proper tensor of rank 2, etc.

The $n = (2\nu + 1)$-dimensional group of rotations $\mathfrak{d}_n$ comprises the $(n-1)$-dimensional one $\mathfrak{d}_{n-1}$ by subjecting the variables $x_1, \cdots, x_{2\nu}$ to an orthogonal transformation and leaving $x_{2\nu+1}$ unchanged. This restriction to a subgroup carries the representation Δ of $\mathfrak{d}_n$, as here defined, over into the representation Δ of the $(n-1)$-dimensional group of rotations which we defined in § 3. The same restriction splits a tensor of rank f in the n-dimensional space into two tensors of rank f and $f-1$ respectively in the $(n-1)$-dimensional space. And thus the decomposition (22) goes over into the decomposition (15).

The matrix C, (17), which satisfied the equations $P'_i = CP_iC^{-1}$ (for $i = 1, 2, \cdots, 2\nu$) fulfills the condition

$$CP_nC^{-1} = (-1)^\nu P'_n$$

for $P_n = P_{2\nu+1}$. Hence it can be used here for the same purpose as in § 4 only if ν even. In the opposite case one must replace C by CP_n:

$$C = \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix} \times \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \times \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix} \times \cdots,$$

and one then has $CP_iC^{-1} = -P'_i$ (for all i). Under both circumstances the equation (18) obtains for the C determined in this manner and after an appropriate normalization of the gauge factor in $S(o)$. Here again we have $\check{\Delta} \sim \Delta$ and we are able to express explicitly the transformation C which changes covariant spinors into contravariant ones.

6. *Splitting of Δ under restriction to proper rotations.* In the case of odd dimensionality it makes no difference whether one considers the group $\mathfrak{d}_n$ or $\mathfrak{d}_n^+$ since the reflection commuting with all rotations is an improper rotation. If, however, $n = 2\nu$ is even, restriction to $\mathfrak{d}_n^+$ effects a splitting of the spin representation Δ into two inequivalent representations Δ^+ and Δ^- of degree $2^{\nu-1}$, and one will have to distinguish between "positive" and "negative" spinors accordingly. This comes about as follows.

Again we form

$$(23) \qquad u = \iota p_1 \cdots p_{2\nu} \rightarrow U = \mathbf{1}' \times \mathbf{1}' \times \cdots \times \mathbf{1}'.$$

We separate the even combinations of signs $(\sigma_1, \cdots, \sigma_\nu)$ as characterized by $\sigma_1 \cdots \sigma_\nu = +1$ from the odd ones. According to such an arrangement U appears in the form

(24)
$$U = \left|\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right|.$$

As a consequence of equations (12) one has for the proper rotations o: $U \to U^* = U$. As $P^*_i = SP_iS^{-1}$ implies $U^* = SUS^{-1}$ the matrix S commutes with (24) and thus breaks up into an "even" and an "odd" part:

$$S = \left|\begin{array}{c|c} S^+ & 0 \\ \hline 0 & S^- \end{array}\right|.$$

The matrices $S^+(o)$ and $S^-(o)$ in the two representations Δ^+ and Δ^- of degree $2^{\nu-1}$ are uniquely determined but for a common sign. Hence the fact that the reflection is associated with the matrix $+1$ in Δ^+, with the matrix -1 in Δ^-, means an actual inequivalence.

What is the significance of the partition of X into four squares for the corresponding quantities x of the algebra Π or for the tensor sets? (1) We see from the equation $UP_i = -P_iU$ that the even quantities commute with U and that the odd ones anticommute. Even and odd quantities are consequently represented by matrices of the following shape respectively:

(25) $\left|\begin{array}{c|c} \times & \\ \hline & \times \end{array}\right|$, (26) $\left|\begin{array}{c|c} & \times \\ \hline \times & \end{array}\right|$

(the squares not marked by a cross are occupied by zeros). (2) The involutorial operation

$$a \to a^* = au, \qquad A \to A^* = AU$$

leaves the two front squares in

$$A = \left|\begin{array}{c|c} + & - \\ \hline + & - \end{array}\right|$$

unchanged while it reverses the signs in the two back squares. Let us agree to ascribe the signature $+$ or $-$ to a quantity a according as $a^* = a$ or $a^* = -a$. These quantities then are represented by matrices of the form (27), (28) respectively:

$$(27)\quad \begin{array}{|c|c|} \hline \times & \\ \hline \times & \\ \hline \end{array}\,, \qquad (28)\quad \begin{array}{|c|c|} \hline & \times \\ \hline & \times \\ \hline \end{array}\,.$$

Every quantity may be uniquely written as the sum of two quantities of signatures + and —. (Besides the operation $a \to a^*$ one could of course also consider the following one: $a \to a^\dagger = ua$. But the crossing of both signatures is carried out in a more convenient way by crossing the signature here applied with the division into even and odd quantities. For we have $a^\dagger = a^*$ for even quantities and $a^\dagger = - a^*$ for odd ones.) Thus we finally get this scheme:

$$\begin{array}{cccc} \begin{array}{|c|c|} \hline \times & \\ \hline & \\ \hline \end{array} & \begin{array}{|c|c|} \hline & \times \\ \hline & \\ \hline \end{array} & \begin{array}{|c|c|} \hline & \\ \hline \times & \\ \hline \end{array} & \begin{array}{|c|c|} \hline & \\ \hline & \times \\ \hline \end{array} \\ \text{even} & \text{odd} & \text{odd} & \text{even} \\ + & - & + & - \;:\ \text{signature.} \end{array}$$

The question as to how our star operation is expressed in terms of tensor sets is answered by the equation:

$$p_1 \cdots p_f \cdot u = (-1)^{\langle f\rangle} \cdot \iota p_{f+1} \cdots p_n,$$

showing that the transition from $a = \{\alpha\}$ to $a^* = \{\alpha^*\}$ is defined by

$$\alpha^*(i'_1 \cdots i'_{n-f}) = (-1)^{\langle f\rangle} \cdot \iota \cdot \alpha(i_1 \cdots i_f)$$

(where $i_1 \cdots i_f\, i'_1 \cdots i'_{n-f}$ is any even permutation). The factor $(-1)^{\langle \nu\rangle} \cdot \iota$ equals i^ν.

Hence, taking into consideration the splitting of $\mathbf{\Gamma}_\nu$ into $\mathbf{\Gamma}_\nu^+ + \mathbf{\Gamma}_\nu^-$ as explained in § 1, we get the following reductions:

$$(29)\qquad \begin{array}{l|l} \Delta^+ \times \breve{\Delta}^+ \sim \Gamma_0 + \Gamma_2 + \cdots & \Delta^+ \times \breve{\Delta}^- \sim \Gamma_1 + \Gamma_3 + \cdots. \\ \Delta^- \times \breve{\Delta}^+ \sim \Gamma_1 + \Gamma_3 + \cdots & \Delta^- \times \breve{\Delta}^- \sim \Gamma_0 + \Gamma_2 + \cdots. \end{array}$$

Of the two sums in the first column, one breaks off with $\mathbf{\Gamma}_{\nu-1}$, the other with $\mathbf{\Gamma}_\nu^+$, whereas the sums of the second column end with $\mathbf{\Gamma}_\nu^-$ and $\mathbf{\Gamma}_{\nu-1}$ respectively.

From (16) we obtain by multiplication

$$(-1)^\nu U' = CUC^{-1} \quad \text{or} \quad CU = (-1)^\nu UC.$$

This shows that C is of form (25) or (26) according as ν is even or odd. With C_1, C_2 being the partial matrices of C, we thus have

$$\left.\begin{array}{ll}\breve{S}^+(o) = C_1\, S^+(o) C_1^{-1}, & \breve{S}^-(o) = C_2\, S^-(o) C_2^{-1} \\ \breve{\Delta}^+ \sim \Delta^+, & \breve{\Delta}^- \sim \Delta^- \end{array}\right\} \quad (\nu \text{ even}),$$

$$\left.\begin{array}{ll}\breve{S}^+(o) = C_1\, S^-(o) C_1^{-1}, & \breve{S}^-(o) = C_2\, S^+(o) C_2^{-1} \\ \breve{\Delta}^+ \sim \Delta^- & \breve{\Delta}^- \sim \Delta^+ \end{array}\right\} \quad (\nu \text{ odd}).$$

7. *Infinitesimal description.*

Even number of dimensions. For the purpose of infinitesimal description it is more convenient to put the quadratic form which is to be left invariant by the orthogonal transformations into the shape

$$x^1y^1 + x^2y^2 + \cdots + x^\nu y^\nu. \tag{30}$$

(x^α, y^α being the $n = 2\nu$ variables). Correspondingly one will have to use the following quantities instead of p_α, q_α:

$$\frac{p_\alpha - iq_\alpha}{2} = s_\alpha, \qquad \frac{p_\alpha + iq_\alpha}{2} = t_\alpha$$

with the relations

$$\begin{array}{lll} s_\alpha t_\alpha + t_\alpha s_\alpha = 1, & s_\alpha t_\beta + t_\beta s_\alpha = 0 & (\text{for } \beta \neq \alpha), \\ s_\alpha s_\beta + s_\beta s_\alpha = 0, & t_\alpha t_\beta + t_\beta t_\alpha = 0 & (\text{for all } \alpha, \beta). \end{array}$$

$$s_\alpha \to S_\alpha = \mathbf{1}' \times \cdots \times \mathbf{1}' \times \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix} \times \mathbf{1} \times \cdots \times \mathbf{1},$$

$$t_\alpha \to T_\alpha = \mathbf{1}' \times \cdots \times \mathbf{1}' \times \begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix} \times \mathbf{1} \times \cdots \times \mathbf{1}.$$

(The factors written down as matrices stand at the α-th place.)

All infinitesimal rotations are linear combinations of rotations of the following types:

(a): $\qquad dx_\alpha = x_\alpha, \quad dy_\alpha = -y_\alpha;$

(b): $\qquad dx_\alpha = x_\beta, \quad dy_\beta = -y_\alpha \qquad (\alpha < \beta).$

(The increments not written down are 0. In (b) one is allowed to exchange independently of each other x_α with y_α and x_β with y_β.) Δ represents (a) by the infinitesimal transformation

$$\tfrac{1}{2}U_\alpha = \tfrac{1}{2}(\mathbf{1} \times \cdots \times \mathbf{1} \times \mathbf{1}' \times \mathbf{1} \times \cdots \times \mathbf{1}) \tag{31}$$

whereas to the infinitesimal rotation (b) corresponds the matrix $S_\alpha T_\beta$. In order to prove this the only thing to be done is to verify the following equations:

(a): $\qquad dX = \tfrac{1}{2}[U_\alpha, X] = \tfrac{1}{2}(U_\alpha X - X U_\alpha) = 0$

for $X = S_\beta$ or T_β $(\beta \neq \alpha)$, but $dS_\alpha = S_\alpha$, $dT_\alpha = -T_\alpha$.

(b):
$$\delta X = [S_\alpha T_\beta, X] = 0 \qquad \text{for all } S \text{ and } T$$

except for $X = S_\beta$ and T_α for which we have:

$$\delta S_\beta = S_\alpha, \qquad \delta T_\alpha = -T_\beta.$$

This is readily seen from the expression

$$[S_\alpha T_\beta, X] = S_\alpha(T_\beta X + XT_\beta) - (XS_\alpha + S_\alpha X)T_\beta.$$

In this way we have arrived at Cartan's infinitesimal description of the spin representation.

Nothing essential has to be added in the case of *odd dimensionality.* It is then most convenient to assume the fundamental quadratic form in the shape

$$(x^0)^2 + 2(x^1y^1 + \cdots + x^\nu y^\nu).$$

(31) shows that Δ *is double-valued and not single-valued.* For in accordance with this equation the rotation o:

$$x^1 \rightarrow e^{i\phi}\, x^1, \quad y^1 \rightarrow e^{-i\phi}\, y^1 \qquad \text{(all other variables unchanged)}$$

is associated with the operation $S(o)$ multiplying the variable $x_{\sigma_1 \ldots \sigma_\nu}$ in the spin space by $e^{\frac{1}{2}i\sigma_1\phi}$ $(\sigma_\alpha = \pm 1)$.

8. *Conditions of reality.* For the *real* orthogonal transformations the question arises whether the conjugate complex representation $\bar{\Delta} : o \rightarrow \bar{S}(o)$ is equivalent to Δ. The P_i being Hermitian matrices, $\bar{P}_i$ equals P'_i. Furthermore, the equations:

$$P^*_i = \sum_k o(ki) P_k \quad \text{imply} \quad \bar{P}^*_i = \sum_k o(ki) \bar{P}_k$$

provided the $o(ik)$ are real. This leads at once to the result

$$\bar{S}(o) = \rho(o) \breve{S}(o).$$

Hence the Hermitian unit form $\Sigma x_A \bar{x}_A$ in spin space goes over, by means of the substitution S, into ρ fold the unit form. So ρ must be positive and

$$|\det S|^2 = \rho^{2^\nu}.$$

But on account of our normalization of S causing $(\det S)^2$ to be $= 1$ we find $\rho = 1$,

$$\bar{S}(o) = \breve{S}(o), \qquad \bar{\Delta} = \breve{\Delta};$$

i. e. *the representation* Δ *of the real orthogonal group is unitary.*

When restricting oneself to real variables one must be aware of the possibility that the fundamental quadratic form

$$\sum_{i,k=1}^{n} a_{ik} x^i x^k \tag{32}$$

may have an *inertial index* t *different from* 0. This is of particular import for physics as, according to relativity theory, $t = 1$ for the four-dimensional world. One now has to subject the determining p_i of the algebra Π to the equation

$$(p_1 x^1 + \cdots + p_n x^n)^2 = \Sigma a_{ik} x^i x^k \quad \text{or} \quad \tfrac{1}{2}(p_i p_k + p_k p_i) = a_{ik}.$$

One will get the new p_i from the old ones by means of the transformation H' if the fundamental form (32) arises from the normal form with $a_{ik} = \delta_{ik}$ by means of the transformation H.

But here again it is convenient to base a more detailed investigation upon the real normal form

$$-(x^1)^2 - \cdots - (x^t)^2 + (x^{t+1})^2 + \cdots + (x^n)^2 = \sum_i \epsilon_i (x^i)^2. \tag{33}$$

(Without any loss of generality we may suppose $2t \leqq n$.) In accordance with physics, let us call the first t variables x^i the temporal, the last $n - t$ the spatial coördinates. The subject of our consideration is the group $\mathfrak{d}_n$ of Lorentz transformations; that is, of all real linear transformations o carrying the fundamental form (33) into itself.†

$P_{t+1}, \cdots, P_n$ keep their previous significance, while $P_1, \cdots, P_t$ assume the factor $i = \sqrt{-1}$. We thus have

$$\bar{P}_i = -P'_i \quad \text{for} \quad (i = 1, \cdots, t); \qquad \bar{P}_i = P'_i \quad \text{for} \quad (i = t+1, \cdots, n).$$

The Hermitian conjugate $\bar{A}'$ of a matrix A may be denoted by $\tilde{A}$. The $\bar{P}_i$ as well as the P'_i satisfy the fundamental rules of commutation. Both sets of matrices must be changed one into the other by means of a certain transformation B. It is easy enough to write down B explicitly:

$$B = i^{t - \langle t \rangle} \cdot P_1 \cdots P_t. \tag{34}$$

To be exact, we have

† To be quite definite: the variables x^i are subjected to the Lorentz-transformation $o : x^i \to \sum_k o(ik) x^k$. The p_i (or P_i) then undergo the contragredient transformation; but in raising the index by means of $p^i = \epsilon_i p_i$ one may introduce quantities p^i transforming cogrediently with the variables x^i.

$$P'_i = B\bar{P}_i B^{-1} \quad \text{or} \quad -P'_i = B\bar{P}_i B^{-1} \tag{35}$$

according as t is even or odd. The factor $i^{t-\langle t\rangle}$ has been added in order to make B Hermitian: $\check{B} = B$. The transposed matrix B' coincides with B but for the sign, namely $B' = (-1)^{\langle t\rangle}B$. In the case of an even n the matrix B is of form (25) or (26) according as t is even or odd. All these properties could be fairly easily derived from general considerations; it is not worth the trouble, however, as one may read them at once from the explicit expression (34).

One obtains from (35) the relation

$$B\bar{S}(o)B^{-1} = \rho(o)\check{S}(o) \tag{36}$$

or after multiplication by $S'(o)$ on the left:

$$S'B\bar{S} = \rho B:$$

the Hermitian form B goes over, by means of the transformation $\bar{S}$, into the multiple ρ of itself. In consequence ρ is real and one infers, in the same manner as in the definite case, the equation

$$\rho(o) = \pm 1.$$

As to its dependence on o, $\rho(o)$ satisfies the condition

$$\rho(o'o) = \rho(o')\rho(o).$$

A new consideration, however, is required for determining this sign ρ. In a Lorentz transformation $\| o(ik) \|$ the temporal minor of the whole determinant:

$$\Omega = \begin{vmatrix} o(11), & \cdots, & o(1t) \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ o(t1), & \cdots, & o(tt) \end{vmatrix} \quad \text{is either } \geqq 1 \text{ or } \leqq -1. \tag{37}$$

We shall put $\sigma_-(o) = +1$ or -1 according as the first or the second case prevails, and call $\sigma_-(o)$ the *temporal signature*; it is a character, i. e.

$$\sigma_-(o'o) = \sigma_-(o') \cdot \sigma_-(o).$$

We need not trouble to prove this here directly because we shall see in the course of our further investigations that the $\rho(o)$ in (36) coincides with $\sigma_-(o)$. In the same manner one may introduce a *spatial signature* $\sigma_+(o)$ by means of the spatial minor of the matrix $\| o(ik) \|$. The latter, though, is $= \sigma(o) \cdot \Omega$;

hence the character $\sigma(o)$ distinguishing the proper and improper transformations equals $\sigma_+\sigma_-$. Of the Lorentz transformations having $\sigma_- = -1$ one may say that they reverse the sense of time whereas those having $\sigma_+ = -1$ reverse the spatial sense. The group of Lorentz transformations falls apart into four pieces not connected with each other and distinguished from each other by the values of the two signatures σ_- and σ_+.

To prove (37) let us introduce the two vectors

$$\mathfrak{o}_i' = \{o(i1), \cdots, o(it)\}, \qquad \mathfrak{o}_i'' = \{o(i, t+1), \cdots, o(in)\}$$

in the realms of the temporal and spatial coördinates respectively. The scalar product $(\mathfrak{a}' \cdot \mathfrak{b}')$ in these two partial spaces has its usual significance $a'_1b'_1 + \cdots + a'_tb'_t$. The relations characteristic for the Lorentz transformation then read:

$$(\mathfrak{o}_i'\mathfrak{o}_k') = \delta_{ik} + (\mathfrak{o}_i''\mathfrak{o}_k'') \qquad (i, k = 1, 2, \cdots, t).$$

From these we derive

$$\begin{vmatrix} (\mathfrak{o}_1'\mathfrak{o}_1'), & \cdots, & (\mathfrak{o}_1'\mathfrak{o}_t') \\ \cdot & \cdots & \cdot \\ (\mathfrak{o}_t'\mathfrak{o}_1'), & \cdots, & (\mathfrak{o}_t'\mathfrak{o}_t') \end{vmatrix} = \begin{vmatrix} 1 + (\mathfrak{o}_1''\mathfrak{o}_1''), & (\mathfrak{o}_1''\mathfrak{o}_2''), & \cdots, & (\mathfrak{o}_1''\mathfrak{o}_t'') \\ \cdot & \cdot & \cdots & \cdot \\ (\mathfrak{o}_t''\mathfrak{o}_1''), & (\mathfrak{o}_t''\mathfrak{o}_2''), & \cdots, & (1 + \mathfrak{o}_t''\mathfrak{o}_t'') \end{vmatrix}$$

$$= 1 + (1/1!) \sum_{i=1}^{t} (\mathfrak{o}_i''\mathfrak{o}_i'') + (1/2!) \sum_{i,k=1}^{t} \begin{vmatrix} (\mathfrak{o}_i''\mathfrak{o}_i'') & (\mathfrak{o}_i''\mathfrak{o}_k'') \\ (\mathfrak{o}_k''\mathfrak{o}_i'') & (\mathfrak{o}_k''\mathfrak{o}_k'') \end{vmatrix} + \cdots.$$

All terms on the right side are $\geqq 0$; hence the whole determinant on the left is $\geqq 1$. This determinant however is the square of Ω.

The fact that the sign ρ *in* (36) *equals* σ_- is proved in the following manner. In accordance with

$$P^*_i = \sum_{k=1}^{n} o(ki)P_k$$

we find

$$(38) \qquad P^*_1 \cdots P^*_t = \begin{vmatrix} o(11) & \cdots & o(1t) \\ \cdot & \cdots & \cdot \\ o(t1) & \cdots & o(tt) \end{vmatrix} \cdot P_1 \cdots P_t + \cdots.$$

But a product like $P_{i_1} \cdots P_{i_t} \cdot P_1 \cdots P_t$ where $i_1 \cdots i_t$ are different indices always has the trace 0 except if $i_1 \cdots i_t$ is a permutation of $1 \cdots t$; whereas

$$\operatorname{tr}(P_1 \cdots P_t \cdot P_1 \cdots P_t) = (-1)^{\langle t \rangle} \operatorname{tr}(P_1{}^2 \cdots P_t{}^2) = (-1)^{t - \langle t \rangle} \cdot 2^\nu.$$

Hence on multiplying equation (38) by $P_1 \cdots P_t$ to the right and forming the trace, one is led to this value of the determinant Ω:

$$2^\nu \Omega = (-1)^{t - \langle t \rangle} \operatorname{tr}(P^*_1 \cdots P^*_t \cdot P_1 \cdots P_t).$$

Using the definitions of S: $P^*{}_i = SP_iS^{-1}$, and of B, one readily obtains:

$$2^\nu\Omega = \mathrm{tr}(SBS^{-1}\cdot B) = \mathrm{tr}(B\cdot SBS^{-1}).$$

According to (36)

$$S^{-1} = \rho B'^{-1}\tilde{S}B' = \rho B^{-1}\tilde{S}B.$$

Replacement of B' by B is allowed as B' coincides with B but for a numerical factor. So one finally gets, with $T = BS = \| t_{JK} \|$:

$$2^\nu\Omega = \rho\cdot\mathrm{tr}(BS\tilde{S}B) = \rho\cdot\mathrm{tr}(BS\cdot\tilde{S}\tilde{B}) = \rho\cdot\mathrm{tr}(T\cdot\tilde{T}) = \rho\cdot\sum_{J,K} |\, t_{JK}\,|^2,$$

and this equation shows ρ to have the sign of Ω.

Any representation $\Gamma : o \rightarrow G(o)$ of the Lorentz group gives rise to another one $\sigma_-\Gamma : o \rightarrow \sigma_-(o)G(o)$. Equation (36) or

$$\bar{S}(o) = \sigma_-(o)B^{-1}\check{S}(o)B$$

then proves the equivalence:

$$\bar{\Delta} \sim \sigma_-\check{\Delta}. \tag{39}$$

The transformation B changes the conjugate of a covariant spinor ψ into a contravariant spinor ϕ: $\phi' = B\bar{\psi}$ (in so far as we confine ourselves to Lorentz's transformations of temporal signature $\sigma_- = 1$). (39) yields, on account of (15), (22), the decompositions

$$(40)\quad \Delta\times\bar{\Delta} \sim \left\{\begin{matrix}\sigma_-\Gamma_0 + \sigma_-\Gamma_1 + \cdots + \sigma_-\Gamma_{\nu-1} + \\ \sigma_+\Gamma_0 + \sigma_+\Gamma_1 + \cdots + \sigma_+\Gamma_{\nu-1}\end{matrix}\right\} + (\sigma_-\Gamma_\nu \sim \sigma_+\Gamma_\nu) \quad [n = 2\nu];$$

$$\Delta\times\bar{\Delta} \sim \sigma_-\Gamma_0 + \sigma_+\Gamma_1 + \sigma_-\Gamma_2 + \cdots \qquad [n = 2\nu + 1].$$

The latter series breaks off with $\sigma_-\Gamma_\nu$ or $\sigma_+\Gamma_\nu$.

In the case $n = 2\nu$ we have the splitting of Δ into Δ^+ and Δ^-, when restricting ourselves to the group $\mathfrak{d}_n{}^+$ of proper Lorentz transformations $[\sigma(o) = 1]$. This restriction wipes out the difference between the two signatures σ_- and σ_+. As we mentioned before, B is of form (25) or (26) according as t is even or odd. Hence one has

for even t: $\quad \bar{\Delta}^+ \sim \sigma_-\check{\Delta}^+, \quad \bar{\Delta}^- \sim \sigma_-\check{\Delta}^-;$
for odd t: $\quad \bar{\Delta}^+ \sim \sigma_-\check{\Delta}^-, \quad \bar{\Delta}^- \sim \sigma_-\check{\Delta}^+.$

9. *Irreducibility. Irreducibility of* Γ_f is granted *a fortiori* if one is able to prove that there does not exist any homogeneous linear relation with constant coefficients (independent of o) among the minors of order f of the matrix of an arbitrary rotation $\| o(ik) \|$. This can be shown without using

any other rotations than permutations of the coördinate axes combined with changes of signs. For let us assume that we have such a non-trivial relation R in which a definite minor $A\binom{i_1 \cdots i_f}{k_1 \cdots k_f}$ occurs with a coefficient different from 0. By suitable exchange we can place this minor in the left upper corner of the matrix. We will now take into account the changes of signs only:

$$\| o(ik) \| = \left\| \begin{matrix} \pm 1 & & & & \\ & \pm 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \pm 1 \end{matrix} \right\|$$

the matrices of which have only their chief minors $A(i_1 \cdots i_f)$ different from 0. The linear relation R will contain, apart from $A(1\,2 \cdots f)$, at least one more term $A(1'\,2' \cdots f')$ with a coefficient different from zero. At least one of the indices $1'\,2' \cdots f'$, let us say l, is different from $1, 2, \cdots, f$. By changing the sign of the one variable x_l, the relation R is carried over into a new one R' in which $A(1\,2 \cdots f)$ occurs with the same, $A(1'\,2' \cdots f')$ however with the opposite coefficient. Hence the sum $\frac{1}{2}(R + R')$ certainly is shorter than R, that is, contains less terms than R; but $A(1\,2 \cdots f)$ occurs in it with the same coefficient different from 0 as before. The procedure of shortening may be continued until the presupposed linear relation $R = 0$ leads to the impossible equation $A(1\,2 \cdots f) = 0$.

These considerations were based upon the *complete* group $\mathfrak{d}_n$. If one allows proper rotations only, $\mathfrak{d}_n^+$, one may have to combine the permutation in the first step with a change of sign of one variable. The second step can be performed in the same manner provided $2f < n$, for then one may choose l as above: as one of the indices $1', 2', \cdots, f'$ different from $1, 2, \cdots, f$, furthermore choose m as an index that does not occur in the row $1, 2, \cdots, f$, $1', 2', \cdots, f'$, and then change the signs of both variables x_l and x_m simultaneously. Even when $n = 2\nu$, $f = \nu$ the procedure of shortening will work as long as the relation R still contains a term $A(1'\,2' \cdots \nu')$ the indices of which are not just the complement $\nu + 1, \cdots, n$ of $1, \cdots, \nu$. Thus one will be led in this case finally to a relation of the form:

$$c\,A(1, 2, \cdots, \nu) + c'\,A(\nu + 1, \cdots, n) = 0. \tag{41}$$

Such a relation obtains indeed:

$$A(\nu + 1, \cdots, n) = A(1, 2, \cdots, \nu)$$

but there exists of course no other one of the type (41). From this we learn not only that the two representations Γ_ν^+ and Γ_ν^- are irreducible, but at the same time that they are *inequivalent*; for it proves that there does not hold any linear relation with fixed coefficients between the components of the two matrices associated with the same arbitrary rotation o in these representations. For the components of these two matrices are

$$\tfrac{1}{2}\left[B\binom{i_1\cdots i_\nu}{k_1\cdots k_\nu} \pm i^\nu B\binom{i'_1\cdots i'_\nu}{k_1\cdots k_\nu}\right]$$

with

$$B\binom{i_1\cdots i_\nu}{k_1\cdots k_\nu} = \tfrac{1}{2}\left[A\binom{i_1\cdots i_\nu}{k_1\cdots k_\nu} + A\binom{i'_1\cdots i'_\nu}{k'_1\cdots k'_\nu}\right].$$

$i_1\cdots i_\nu, i'_1\cdots i'_\nu$ and $k_1\cdots k_\nu, k'_1\cdots k'_\nu$ are even permutations of the figures $1, 2, \cdots, n$. The reasoning above shows that there exists no universal linear relation between the quantities $B\binom{i_1\cdots i_\nu}{k_1\cdots k_\nu}$.

The *inequivalence* of two such Γ_f the ranks f of which do not give the sum n, is granted by their having different degrees.

This whole argument was based upon the *complex* orthogonal group. But nothing is to be modified when one confines oneself to the *real* orthogonal transformations. Furthermore one sees, by formulating the result in an infinitesimal manner, that it cannot be effected by the inertial index. The infinitesimal transformation

$$(42) \qquad dx_i = x_k, \quad dx_k = -x_i \qquad (i \neq k)$$

(all other increments being 0; this transformation engenders the permutation $x_i \to x_k$, $x_k \to -x_i$ as well as the change of sign $x_i \to -x_i$, $x_k \to -x_k$) has to be replaced, if the fundamental quadratic form contains terms with the minus sign, for couples (x_i, x_k) consisting of a temporal and a spatial variable by

$$dx_i = x_k, \quad dx_k = x_i$$

while it has to be kept unchanged for couples of variables (x_i, x_k) both temporal or both spatial. The statement of irreducibility under all transformations (42) in the definite case is identical with the statement of irreducibility under the transformations replacing them in the indefinite case; one only needs to replace the temporal variables x_k by $\sqrt{-1}\cdot x_k$.

The product $\Gamma \times \check{\Gamma}$ of a representation Γ with its contragredient $\check{\Gamma}$ contains the identity Γ_0 at least μ times when Γ reduces into μ parts. If we are allowed to make use of the general and elementary theorem that the irreducible

parts of a representation are uniquely determined † (in the sense of equivalence and except for their arrangement), then the formulae (15), (22), (29) show at once the irreducibility of Δ or Δ^+ and Δ^- respectively and the inequivalence of the latter. Another direct proof runs as follows:

Take the full group $\mathfrak{d}_n$ in the even case $n = 2\nu$. Using the fundamental quadratic form in the shape (30), let us consider the "diagonal" infinitesimal rotations

$$dx_\alpha = i\phi_\alpha x_\alpha, \qquad dy_\alpha = -i\phi_\alpha y_\alpha \qquad (\alpha = 1, \cdots, \nu) \tag{43}$$

(ϕ_α independent parameters). It is associated in Δ with the diagonal transformation

$$dx_{\sigma_1 \ldots \sigma_\nu} = (i/2)(\sigma_1\phi_1 + \cdots + \sigma_\nu\phi_\nu) x_{\sigma_1 \ldots \sigma_\nu} \qquad (\sigma_\alpha = \pm).$$

Given a partial space P' of the total spin space P, different from 0 and invariant under Δ, one chooses a non-vanishing vector z:

$$z = \sum_A z_A e_A = \{z_A\} \qquad [A = (\sigma_1, \cdots, \sigma_\nu)]$$

occurring in P'. By performing the substitution (43) repeatedly one is able to isolate each term $z_A e_A$, as these parts are of different "weights" $(i/2)(\sigma_1\phi_1 + \cdots + \sigma_\nu\phi_\nu)$. Therefore at least one of the fundamental vectors e_A occurs in P'. But $e_A = e_{\sigma_1 \ldots \sigma_\nu}$ goes over into any other fundamental vector $e_{\tau_1 \ldots \tau_\nu}$ by exchanging $x_\alpha \to y_\alpha$, $y_\alpha \to x_\alpha$ those couples (x_α, y_α) for which the signs σ_α and τ_α do not coincide. P' is therefore identical with the total P.—Irreducibility of Δ for odd $n = 2\nu + 1$ is an immediate consequence of the irreducibility for even n, we just proved; one has to restrict oneself merely to the subgroup $\mathfrak{d}_{n-1}$ within $\mathfrak{d}_n$, $n = 2\nu + 1$. One sees in the same manner that the two parts Δ^+, Δ^- are irreducible and inequivalent for the group $\mathfrak{d}_n^+$, $n = 2\nu$.

10. *Dirac's theory.* Let us suppose we are dealing with a *spinor field* $\psi^A(x^1 \cdots x^n)$ in an n-dimensional "world" with the fundamental metric form (33). The most essential feature of *Dirac's theory* is that one should be able to form a *vector* by linear combination of the products $\bar{\psi}^A\psi^B$. If n is even, one sees from equation (40) that exactly *one* such vector s_i exists—that behaves like a vector at least for all Lorentz transformations not reversing the sense of time; and *one* such vector for all Lorentz transformations not reversing the spatial sense. In the case n odd, one vector of the second, and

† Compare e. g. Weyl, *Theory of Groups and Quantum Mechanics* (London, 1931), p. 136.

no vector of the first kind exists. Only the first type can be used when one believes in the equivalence of right and left, but is prepared to abandon the equivalence of past and future. n has then to be even and the vector is

$$s_i = \tilde{\psi} B P_i \psi.$$

From this vector one can derive the scalar field:

$$(44) \qquad \sum_i \tilde{\psi} B P^i (\partial\psi/\partial x^i) \qquad (P^i = \epsilon_i P_i).$$

One needs a scalar that arises from linear combination of the products $\tilde{\psi}^A \cdot \partial\psi^B/\partial x^i$ in Dirac's theory as the main part of the *action quantity* which accounts for the fundamental features of the whole quantum theory. There is no ambiguity: for $(\Delta \times \bar{\Delta}) \times \breve{\Gamma}_1$ contains the identity Γ_0 or rather the representation $\sigma_- \Gamma_0$ just once if decomposed into its irreducible parts. That is shown by equation (40) when one takes into account the fundamental lemma of the theory of representations asserting that the product $\Gamma \times \breve{\Gamma}_1$ contains the identity Γ_0 once, or not at all, according as the two irreducible representations Γ, Γ_1 of the same group are equivalent or not. Dirac's quantity of action contains, apart from (44), a second term which is a linear combination of the undifferentiated products $\tilde{\psi}^A \psi^B$; it is multiplied by the mass, and accounts for the inertia of matter. There exists just one such scalar, namely $\tilde{\psi} B \psi$, in the case of an even as well as an odd n.

Furthermore one may consider as essential the fact that the time component of the electric current is positive-definite in Dirac's theory, namely proportional to the "probability density" $\sum_A \bar{\psi}^A \psi^A$; this grants the atomistic structure of electric charge. If the fundamental form (33) is of inertial index t, this property however is not possessed by the vector contained in $\Delta \times \bar{\Delta}$ but by the tensor of rank t with the components

$$s_{i_1 \ldots i_t} = \tilde{\psi} B P_{i_1} \cdots P_{i_t} \psi \qquad (i_1, \cdots, i_t \text{ different}),$$

the "temporal" component, $s_{12 \ldots t}$, of which is $= \bar{\psi}\psi$ (but for a numerical factor). It seems to be required by the scheme of Maxwell's equations that electric current should be a vector; this requirement, together with the postulate of the atomic structure of electricity, compels us to assume the inertial index t to be $= 1$.

11. *Appendix. Automorphisms of the complete matrix algebra. A one-*

to-one correspondence $X \rightleftarrows X^*$ *of the ring of all n-rowed matrices upon itself is isomorphic when satisfying the conditions*

$$(X+Y)^* = X^* + Y^*, \qquad (\lambda X)^* = \lambda \cdot X^*, \qquad (XY)^* = X^* Y^*$$

(λ *an arbitrary number*). *The only such automorphism is "similarity":*

$$X^* = AXA^{-1},$$

A being a fixed non-singular matrix.

Proof. The equation $GX = \gamma X$ has a solution $X \neq 0$ only if γ is an eigen-value of the matrix G; for the columns of the matrix X must be eigen-vectors belonging to the eigen-value γ. The eigen-values of G thus are characterized in a manner invariant with respect to the given automorphism. Consequently G^* has the same eigen-values as G. Thus we are led to proceed as follows. Let us choose n fixed different numbers $\gamma_1, \cdots, \gamma_n$ and with them form the diagonal matrix

$$G = \left\| \begin{matrix} \gamma_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \gamma_n \end{matrix} \right\|.$$

As G^* has the same eigen-values as G, a non-singular matrix A can be determined such that $G^* = AGA^{-1}$. Let us replace every X^* by $X^{**} = A^{-1}X^*A$ and now consider the automorphism $X \to X^{**}$ that leaves G unchanged. The matrix E_{ik} containing an element different from 0, namely 1, only at the crossing point of the i-th row with the k-th column is determined by the properties

$$GE_{ik} = \gamma_i E_{ik}, \qquad E_{ik}G = \gamma_k E_{ik}$$

except for a numerical factor. Hence we have

$$E_{ik} \to E^{**}_{ik} = \alpha_{ik} E_{ik}. \tag{45}$$

The equation $E^2_{ii} = E_{ii}$ furnishes $\alpha^2_{ii} = \alpha_{ii}$, $\alpha_{ii} = 1$. After putting $\alpha_{i1} = \alpha_i$, $\alpha_{1k} = \beta_k$, the relation

$$E_{ik} = E_{i1}E_{1k}$$

leads to $\alpha_{ik} = \alpha_i \beta_k$. On account of $\alpha_{ii} = 1$ one therefore has $\beta_i = 1/\alpha_i$ and $\alpha_{ik} = \alpha_i/\alpha_k$. Hence in accordance with (45) an arbitrary matrix $X = \| x_{ik} \|$ and its image $X^{**} = \| x^{**}_{ik} \|$ are linked by the relation

$$x_{ik}^{**} = \alpha_i x_{ik}/\alpha_k \quad \text{or} \quad X^{**} = A_0 X A_0^{-1}$$

where A_0 is the diagonal matrix with the terms $\alpha_1, \cdots, \alpha_n$.

This demonstration furnishes a method for constructing a spinor from a given tensor set g. The method will be used preferably in the case where g consists of only one tensor of definite rank. Our representation of degree 2^ν of the algebra Π associates with g a matrix G. Let us assume that G has the (simple) eigen-value γ and let ψ be the corresponding eigen-vector in spin space: $G\psi = \gamma \cdot \psi$. The rotation o carries g into a set $g(o)$ represented by the matrix $G(o)$. γ is a (simple) eigen-value of $G(o)$ as well as of G, and the solution $\psi(o)$ of the equation

$$G(o)\psi(o) = \gamma \cdot \psi(o)$$

arises from ψ by the transformation $S(o)$ corresponding to o in the spin representation.

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ON THE THEORY OF APPORTIONMENT.

By William R. Thompson.

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1. If in an accepted sense, P is the probability that one method of treatment, T_1, is better than a rival, T_2, we may develop a system of apportionment such that the proportionate use of T_1 is $f_{(P)}$, a monotone increasing function, rather than make no discrimination at all up to a certain point and then finally entirely reject one or the other. The only paper * which has so far appeared in his field, as far as I am aware, is one by myself in a recent issue of *Biometrika*. In this paper I have considered the case of choice between two such rival *treatments*,† and for symmetry suggested that $f_{(Q)} \equiv 1 - f_{(P)}$ where $Q = 1 - P$. Then the *risk* of assignment to T_1 when it is not the better is $Q \cdot f_{(P)}$, while the corresponding risk for T_2 is $P \cdot f_{(Q)}$. Accordingly, I suggested further that we set $f_{(P)} = P$, which is a necessary and sufficient condition that these two risks be equal. Their sum, the *total risk*, is then $2PQ$.

A special case was considered wherein the result of use of T_i at any given trial is either *success* or *failure*, the probability of failure being an unknown, p_i, *à priori* (independently for $i = 1, \cdots, k$) equally likely to lie in either of any two equal intervals in the possible range, $(0, 1)$. It is further assumed that for a given T_i we have an experience of exactly n_i independent trials, the number of *successes* being s_i and of *failures* being $r_i \equiv n_i - s_i$; and the probability of obtaining such a sample is

$$\binom{n_i}{r_i} \cdot p_i^{r_i} \cdot q_i^{s_i} \text{ where } q_i = 1 - p_i.$$

Restricting consideration to the case, $k = 2$, dropping the subscript *one* and using a prime instead of subscript *two*, then it was shown that

$$(1) \qquad P = \psi_{(r,s,r',s')} \equiv \frac{\sum\limits_{\alpha=0}^{r'} \binom{r + r' - \alpha}{r} \cdot \binom{s + s' + 1 + \alpha}{s}}{\binom{n + n' + 2}{n + 1}}.$$

Now, it is well known that the probability, $\bar{P}$, that by drawing at random

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* W. R. Thompson, *Biometrika*, vol. 25 (1933), pp. 285-294.

† By *treatment* we imply a special mode of dealing with *individuals* of a given class of things.

without replacements from a mixture of W *white* and B *black* balls we shall encounter w white before b black is given by

$$(2) \qquad \bar{P} = \frac{\sum_{\alpha=0}^{h} \binom{W}{w+\alpha} \cdot \binom{B}{b-1-\alpha}}{\binom{W+B}{w+b-1}},$$

where $h = \mathrm{Min}(b-1, W-w)$. The object of the present paper is *first*, to show exactly how ψ may be expressed in the form of (2) and thus make possible the use of a machine based on this principle in the apportionment, and thereby avoid an enormous amount of calculation where tables are not available; and *second*, to develop a complete statement of the group, G, of substitutions of the arguments of $\psi_{(a_1,a_2,a_3,a_4)}$ which leave ψ invariant, and also those of the set, A, which change the value to $1-\psi_{(a_1,a_2,a_3,a_4)}$. The application of these substitutions to give a convenient form for calculation * of ψ or for other purposes is obvious. On this account the ψ-function is a convenient form for expression † of the incomplete hypergeometric series, as in the case of two problems considered by Pearson,‡ where for certain original variables which we may denote by a, b, c, and d we may express § a required probability by $\psi_{(a,b,c,d-1)}$.

2. We begin by considering the function, $\bar{N}_{(r,s,r',s')}$ of four rational integers $\geqq 0$, defined by

$$(3) \qquad \bar{N}_{(r,s,r',s')} \equiv \sum_{\alpha=0}^{\alpha \leqq s,r'} \binom{r+r'+1}{r+1+\alpha} \cdot \binom{s+s'+1}{s-\alpha},$$

and extend this definition to include

$$(4) \qquad \bar{N}_{(r,s,-1,s')} \equiv 0 \equiv \bar{N}_{(r,-1,r',s')}, \quad \text{and}$$
$$\bar{N}_{(r,s,r',-1)} \equiv \binom{r+s+r'+1}{r'} \equiv \bar{N}_{(-1,r',s,r)}.$$

Now, in the previous paper,¶ I have defined an *N-function* identical with $\bar{N}$ for the arguments in (4) and otherwise equal to the numerator of the right member of (1). Obviously,

* B. H. Camp, *Biometrika*, vol. 17 (1925), pp. 61-67.

† W. R. Thompson, *loc. cit.*

‡ Karl Pearson, *Philosophical Magazine*, Series 6, vol. 13 (1907), pp. 365-378; *Biometrika*, vol. 20A (1928), pp. 149-174.

§ W. R. Thompson, *loc. cit.*

¶ W. R. Thompson, *loc. cit.*

$$\bar{N}_{(r,s,r',s')} \equiv \bar{N}_{(s',r',s,r)} \equiv \binom{n+n'+2}{n+1} - \bar{N}_{(s,r,s',r')},$$

as has been proved for the N-function,* and

(5) $$\bar{N}_{(r,s,0,s')} \equiv N_{(r,s,0,s')};$$

and we may verify readily by (3) that *in general*

(6) $$\bar{N}_{(r,s,r',s')} \equiv \bar{N}_{(r,s-1,r',s')} + \bar{N}_{(r,s,r',s'-1)},$$

which relation was shown in my first paper to hold for the N-function also. Accordingly, by complete induction we may demonstrate that

(7) $$\bar{N}_{(r,s,r',s')} \equiv N_{(r,s,r',s')},$$

and therefore

(8) $$\psi_{(r,s,r',s')} \equiv \frac{\sum_{\alpha=0}^{\alpha\leq s,r'} \binom{r+r'+1}{r+1+\alpha} \cdot \binom{s+s'+1}{s-\alpha}}{\binom{r+s+r'+s'+2}{r+s+1}},$$

By a simple rearrangement of factors after expressing the binomial coefficients in (8) by factorial numbers we may obtain

(9) $$\psi_{(r,s,r',s')} \equiv \frac{\sum_{\alpha=0}^{\alpha\leq s,r'} \binom{r+s+1}{r+1+\alpha} \cdot \binom{r'+s'+1}{r'-\alpha}}{\binom{r+r'+s+s'+2}{r+r'+1}}$$

which is the equivalent of the expression in (2) if we set $W=r+s+1$, $B=r'+s'+1$, $w=r+1$ and $b=r'+1$, which is the required relation. Furthermore, (8) and (9) give

(10) $$\psi_{(r,s,r',s')} \equiv \psi_{(r,r',s,s')},$$

i. e., $\psi_{(a_1,a_2,a_3,a_4)}$ is invariant under the substitution $(2,3)$, which therefore belongs to the group, G. Now, by the identities of (10) and (23) of the previous paper,† we have obviously established that $(1,4)(2,3)$ is also in G, and that $(1,2)(3,4)$ changes ψ to $1-\psi$ and is therefore an element of the set A. On the other hand if $a_1=3$, $a_2=2$, $a_3=1$, and $a_4=0$, the substitution $(1,3)$ brings a change in value of ψ from 9/14 to 13/14, and therefore $(1,3)$ belongs neither to G nor A. Now, if the four arguments are all different they may be arranged in 24 different ways; whence, if m is the

* W. R. Thompson, *loc. cit.*

† W. R. Thompson, *loc. cit.*

number of different substitutions in the group, G, then $24/4 \geqq m \equiv 0 \pmod 4$. Accordingly, we have established the fact that the complete group leaving ψ invariant is generated by the two transpositions, $(2,3)$, and $(1,4)$; i. e.,

$$G = [(2,3), (1,4)]. \tag{11}$$

Moreover, the set of substitutions, A, changing ψ to $1-\psi$ may be represented in the form,

$$A = \{g \cdot (1,2)(3,4)\} \tag{12}$$

where g is an element of G.

By the aid of (11) and (12) we may prove and state in simple form certain relations,* and prior to any use of the ψ-function obtain the most convenient arrangement for the work; and in tabulations only 3 values need be listed for each combination of the four arguments without loss of completeness, namely $\psi_{(a,b,c,d)}$, $\psi_{(a,b,d,c)}$, and $\psi_{(a,c,d,b)}$. We may readily verify also that if two of these arguments are equal then two of the three values are sufficient, if three of the arguments are equal or there are two pairs of equal arguments then one value is enough, and if $a = b = c = d$ then none is needed in order to evaluate ψ in a simple manner by means of (11) and (12). By use of the N-function as previously suggested † instead of ψ intabulation in a systematic process with increasing arguments we may list only values of this reduced form of table; e. g., $a \geqq d \geqq c \geqq b > 0$ with the relations given in (6) and (7) and

$$N_{(r,s,r',0)} \equiv \binom{r+s+r'+2}{r'+1} - \binom{r+r'+1}{r} \tag{13}$$
$$\text{and} \quad N_{(r,0,r',s')} \equiv \binom{r+r'+1}{r}.$$

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* W. R. Thompson, *loc. cit.*

† Thus we may obtain readily, the relation,

$$\psi_{(r,s,r',s')} \equiv \psi_{(r-1,s,r',s')} - \frac{(s+s'+1)\cdot\binom{r+r'}{r}\cdot\binom{s+s'}{s}}{(r+s+1)\binom{r+s+r'+s'+2}{r+s+1}},$$

and simply from limit relations previously established,

$$I_{p(r+1,s+1)} \equiv I_{p(r,s+1)} - \binom{n}{s} p^r \cdot q^{s+1}$$
$$\equiv I_{p(r+1,s)} + \binom{n}{r} p^{r+1} \cdot q^s$$

where $q = 1 - p$, and $I_{x(u,v)} \equiv \dfrac{B_{x(u,v)}}{B_{1(u,v)}}$.

3. For my own purposes I constructed a rough machine based on the probability relation (9) as follows:

I took the cover of a square cardboard box, which I cut and bent along the diagonal forming a box having the shape of an isosceles triangle with 45° base angles. In this I placed $n + n' + 2$ balls as used in bearings. Of these $n' + 1$ had been made dull by a copper sulphate bath. I shall call these *black* and the others *white*. I then shuffled these balls in the box, and at random allowed them all * to line up along the long side or hypotenuse of the box. This alignment I regarded as a draft proceeding from left to right. Here the advantage of a prior arrangement of the arguments of ψ so as to make the number of balls to be scanned as small as possible is apparent. The critical condition was to encounter $r + 1$ *white* before $r' + 1$ *black* balls.

I supposed now that I was considering a case of the sort where I have to assign individuals to one of two methods of treatment,† T_1 and T_2, in proportion based on the ψ-function of the accumulated evidence in the conventional r, s, r', s' form. I then gave certain values to p_1 and p_2 to govern the chance of *failure* when T_1 and T_2 were tried, respectively; but otherwise acted as if p_1 and p_2 were unknown. Starting with no experience, then $r = s = r' = s' = 0$, I placed $r + s + 1 = 1$ white and $r' + s' + 1 = 1$ black ball in the box, and shuffled. After alignment then T_1 was chosen if the white ball was at the left and otherwise T_2 was chosen. The treatment chosen, T_i, was *tried* by the corresponding probability, p_i, and the result recorded in new values of r, s, r', s'; i. e., if T_2 were tried with *success* these new values then were $0, 0, 0, 1$; if with failure then they would have been $0, 0, 1, 0$. Similar remarks hold if T_1 were chosen. I then added a ball, white if T_1 had been tried and otherwise black. These three balls were now shuffled and aligned at random. As before, if the critical condition of encountering $r + 1$ white before $r' + 1$ black balls were met then the treatment, T_1 was used at this turn, and otherwise T_2. The result of the treatment indicated was noted and new values of r, s, r', s' obtained, and another ball added to the box according to the criterion described for the last turn, and so on until a given number of trials had been made.

In the accompanying table values of p_1 and p_2 used in such experiments are given together with the final results—the total number of trials, $n + n'$;

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* As a matter of fact it is not necessary that all the balls be lined up. The object is simply to quickly establish a random draft order.

† By *treatment* we imply a special mode of dealing with *individuals* of a given class of things.

the number of these wherein the *conventionally* worse method (T_1) was used, n; and the number of *failures*, r, among these n trials.

To make the table quite clear, take the numbers in the second row. Here we have the record of four parallel experiments wherein T_2 was governed by a condition such that failure might be expected about half the time and T_1 to fail always. The total number of trials, $n + n' = 40$, and the number of these systematically allotted to T_1 was $n = 5, 9, 7$, and 5 in the respective experiments, and r, of course, had the same values here. The relatively small value of these even in so small a total number of trials, indicates strikingly the rapidity with which this systematic apportionment between the rival treatments, T_1 and T_2, tends to favor the better, even though prior knowledge as to the fact that T_2 is the better is disregarded.

Although the machine used is extremely crude, all the results obtained were extremely favorable. A more carefully constructed machine along the same lines might give even better results. I have conducted a few additional experiments with this simple box, in which I have deliberately arranged an unfavorable start. I was greatly pleased to note the rapidity with which the machine brought about a reversal of favor to the better method, T_2, as the experiments proceeded.

4. The system of apportionment which we have examined admits a simple extension to the general case of k rival *treatments*, (T_i). As defined in § 1, we let p_i represent the unknown probability of failure by treatment T_i, and our experience with this treatment to consist of r_i failures and s_i successes, where $i = 1, \cdots, k$. Now, if we place $r_i + s_i + 1$ balls of a kind, C_i; for $i = 1, \cdots, k$; in our box, shuffle and draw as before, then we note that the probability of drawing $r_i + 1$ of the i-th *kind* before $r_j + 1$ of the j-th kind is independent of the presence of the balls of other kinds and identical with P_{ij} where $i \neq j$ and

$$P_{ij} \equiv \psi_{(r_i, s_i, r_j, s_j)}. \tag{14}$$

Thus we see that the probability that $r_i + 1$ balls of the i-th kind be so drawn before $r_j + 1$ of the j-th kind, where $i \neq j = 1, \cdots, k$ is *exactly* P_i defined by the relation

$$P_i \equiv \prod_{i \neq j=1}^{k} P_{ij} \equiv 2 \prod_{j=1}^{j=k} P_{ij}. \tag{15}$$

Arbitrarily, as in the case $k = 2$, we may apportion *individuals* among the k rival *treatments* by assigning to each T_i the portion, f_i, or making the chance of this assignment equal f_i, respectively. We may thus arbitrarily take $f_i = P_i$,

which may be calculated or we may use the machine, as we have seen that a unique answer is given at each turn just as in the special case, considered previously. Unlike that case, however, we are unable to state that P_i is the probability that T_i is the best of the k rivals; but its composition in (15) indicates that it may well serve the proposed purpose.

TABLE.

p_1	p_2	Total Trials $(n + n')$	Trials of T_1 (n)	Failures of T_1 (r)	Approx. $(n \cdot p_2)$*
1	0	20	2, 1, 1, 1	2, 1, 1, 1	0
1	1/2	40	5, 9, 7, 5	5, 9, 7, 5	2, 4, 3, 2
1/2	0	40	6, 2, 3, 5	2, 2, 2, 2	0
3/4	1/4	100	3, 4	3, 3	1, 1
1	3/4	100	14, 10	14, 10	10, 7
3/4	1/2	100	23, 14	17, 11	8, 5
1/2	1/4	100	10, 13	5, 6	2, 3
1/4	0	100	4, 6	1, 1	0

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* *Expectation* of loss in the same n had T_2 been used.

ON A GENERALIZED TANGENT VECTOR.*

By H. V. Craig.

1. *Introduction.* The purpose of this paper is: (a) to prove that the left members, E_r, of the Euler equations associated with the function $F(x^1, \cdots, x^n;\ dx^1/dt, \cdots, dx^n/dt; \cdots;\ d^m x^1/dt^m, \cdots, d^m x^n/dt^m)$ and the quantities T_r, to be introduced, transform as the components of covariant tensors; (b) to make manifest certain points of similarity existing between T_r and the covariant tangent vector of Synge-Taylor geometry; and (c) to indicate a rôle that T_r might play in the development of a geometry based on F.

In Section 3 we develop certain formulas based on the rule for differentiating a product and in 4 apply them to establish by induction the covariance of E_r and T_r. These tensors are associated with the function F and the induction consists of proving that if for a given $F(x, \cdots, d^m x/dt^m)$ the T_r and E_r related to $F(x, \cdots, d^{m-1}x/dt^{m-1}, K)$ (K is a set of n constants) are tensors then the same may be asserted of the T_r and E_r corresponding to $F(x, \cdots, d^{m-1}x/dt^{m-1}, d^m x/dt^m)$.

2. *Notation.* The symbolism to be employed in this paper is exhibited in the following table:

$x'^r = dx^r/dt$; $x^{(p)r} = d^p x/dt^p$; ${}_0C_0 = 1$; ${}_mC_u$ is a binomial coefficient;
$\partial x^r/\partial y^i = X_i{}^r = X^{(o)r}_{(o)i}$; $\partial x^{(v)r}/\partial y^{(u)i} = X^{(v)r}_{(u)i}$; $\partial F/\partial x^{(u)r} = F_{(u)r}$;
$\partial \bar{F}/\partial y^{(u)i} = \bar{F}_{(u)i}$; $T_r = \sum_{u=1}^{m} u(-1)^{u-1} F^{\cdot\cdot(u-1)}_{(u)r}$; $E_r = \sum_{u=0}^{m} (-1)^{u+1} F^{\cdot\cdot(u)}_{(u)r}$;
$S_r = F_{(o)r} + \sum_{u=2}^{m} (u-1)(-1)^{u-1} F^{\cdot\cdot(u)}_{(u)r} - T_j\{{}^{j}_{r}\}$.

3. *Preliminary formulae.* The point transformation of $x^r = x^r(y)$ gives rise to the following equalities:

$x'^r = X_i{}^r y'^i$; $x^{(m+1)r} = (X_i{}^r y'^i)^{(m)} = \sum_{u=0}^{m} {}_mC_u X^{(m-u)r}{}_i\, y^{(u+1)i}$;
$F(x, x', \cdots, x^{(m)}) = \bar{F}(y, y', \cdots, y^{(m)})$; $X^{(m+1)r}_{(m+1)j} = X_j{}^r$;
$X^{(m+1)r}_{(m)j} = {}_mC_{m-1} X_j{}^{(1)r} + {}_mC_0 X_j{}^{(1)r} = {}_{m+1}C_1 X_j{}^{(1)r}$.

This last relationship suggests the formula:

$$X^{(m)r}_{(m-l)j} = {}_mC_l X_j{}^{(l)r} \tag{1}$$

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* Presented to the Society, June 20, 1934.

which we shall now proceed to establish by induction, thus

$$X^{(m+1)r}_{(m-l)j} = {}_mC_{m-l-1}X_j{}^{(l+1)r} + \sum_{u=0}^{l} {}_mC_u X^{(m-u)r}_{(m-l)ji} y^{(u+1)i}$$
$$= {}_mC_{m-l-1}X_j{}^{(l+1)r} + \sum_{u=0}^{l} {}_mC_u\, {}_{m-u}C_{l-u} X_{ji}{}^{(l-u)r} y^{(u+1)i}$$
$$= {}_mC_{m-l-1}X_j{}^{(l+1)r} + {}_mC_l \sum_{u=0}^{l} {}_lC_{l-u} X_{ji}{}^{(l-u)r} y^{(u+1)i}$$
$$= {}_{m+1}C_{l+1}X_j{}^{(l+1)r}.$$

A second equality to be used in the sequel is as follows:

$$(2) \qquad \sum_{u=1}^{m} u(-1)^{u-1}(FX^{(m)r}_{(u)i})^{(u-1)} = m(-1)^{m-1}F^{(m-1)}X_i{}^r.$$

To verify this we note that by virtue of (1) the left member of the foregoing may be written

$$\sum_{u=1}^{m} u(-1)^{u-1}\, {}_mC_u \sum_{s=0}^{u-1} {}_{u-1}C_s F^{(s)} X_i{}^{(m-1-s)r}.$$

Now if s is not $m-1$ we have

$$\sum_{u=s+1}^{m} u(-1)^{u-1}\, {}_mC_{m-u}\, {}_{u-1}C_{u-1-s} = m\, {}_{m-1}C_{m-1-s} \sum_{u=s+1}^{m} (-1)^{u-1}\, {}_{m-s-1}C_{u-s-1}$$
$$= \pm m\, {}_{m-1}C_{m-1-s} \sum_{l=0}^{m-s-1} (-1)^l\, {}_{m-s-1}C_l = 0,$$

from which (2) follows.

4. *The vector character of T_r and E_r.* The covariance of T_r for m equal to one or two may be established * readily and so we pass on to the induction. Thus, if for any function $F(x, x', \cdots, x^{(m-1)})$ T_r is a covariant vector, then we may write

$$\sum_{u=1}^{m} u(-1)^{u-1}F_{\cdot\cdot(u)i}^{(u-1)} = \sum_{u=1}^{m-1} u(-1)^{u-1}F_{\cdot\cdot(u)r}^{(u-1)}X_i{}^r \dagger$$
$$+ \sum_{u=1}^{m} u(-1)^{u-1}(F_{(m)r}X^{(m)r}_{(u)i})^{(u-1)}$$

from which we attain our conclusion by way of (2).

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* See H. V. Craig, "On parallel Displacement in a non-Finsler space," *Transactions of the American Mathematical Society*, vol. 33 (1931), p. 133.

† The assumption that the T_r associated with $F(x, x', \cdot\cdot\cdot, x^{(m-1)}, k)$ is a tensor implies that for $F = F(x, x', \cdot\cdot\cdot, x^{(m-1)}, k)$

$$\sum_{u=1}^{m-1} u(-1)^{u-1} \sum_{p=u}^{m-1} (F_{(p)r}X^{(p)r}_{(u)i})^{(u-1)} = \sum_{u=1}^{m-1} u(-1)^{(u-1)}F_{\cdot\cdot(u)r}^{(u-1)}X_i{}^r$$

and from the nature of this reduction it follows that the same simplification can be made if k is replaced with $x^{(m)}$.

This accomplished there remains to be proved that the left members of the Euler equations transform according to a tensor law, or more explicitly that $\bar{E}_i = E_r X_i{}^r$. Again we employ mathematical induction, thus

$$\sum_{u=0}^{m} (-1)^{u+1} \bar{F}_{(u)i}^{\cdot\cdot(u)} = \sum_{u=0}^{m-1} (-1)^{u+1} F_{(u)r}^{\cdot\cdot(u)} X_i{}^r + \sum_{u=0}^{m} (-1)^{u+1} (F_{(m)r} X_{(u)i}^{(m)r})^{(u)}.$$

By expanding the last term of the foregoing and evaluating the derivatives of $x^{(m)r}$ by means of (1) we obtain the expression

$$\sum_{u=0}^{m} (-1)^{u+1} \sum_{s=0}^{u} {}_uC_s\, F_{(m)r}^{\cdot\cdot(s)}\, {}_mC_u X_i{}^{(m-s)r}$$

which reduces to $(-1)^{m+1} F_{(m)r}^{\cdot\cdot(m)} X_i{}^r$ since for s not m

$$\sum_{u=s}^{m} (-1)^{u+1}\, {}_mC_u\, {}_uC_s$$

is zero.

5. *Certain generalized geometries.* A metric manifold such that the arc length of a curve C ($C : x^r = x^r(t)$) is given by the integral $\int F(x^1, \cdots, x^n; x'^1, \cdots, x'^n)\,dt$ is called a Finsler space. The function F is among other things assumed to satisfy the identity $x'^r F_{(1)r} = F$; this insures the invariance under parameter change of the integral $\int F dt$. J. L. Synge and, independently, J. H. Taylor have investigated the geometry of a Finsler space having for its metric tensor the quantities f_{rs} $(2f_{rs} = F^2{}_{(1)r(1)s})$. As an immediate consequence of the identity $x'^r F_{(1)r} = F$ they derive that $x'^s f_{rs} = F F_{(1)r}$. Consequently if the parameter is the Finsler arc (in this case F maintains the value unity along the curve in question) the quantities x'^r, $F_{(1)r}$ are said to be the contravariant and covariant descriptions of the unit tangent vector. One of the salient properties of this geometry is that the auto-parallel curves $\theta x^r = 0$ * coincide with the extremals associated with F. Likewise, it may be proved easily that $\theta F_{(1)r} = E_r$ $(E_r = F'_{(1)r} - F_{(0)r}$. Furthermore, $x'^r \theta F_{(1)r} = 0$ and so the vector θF_{1r} may be regarded as the covariant principal normal vector.

Spaces involving metric tensors whose components are functions of not only x and x' but of higher derivatives as well were first investigated by Akitsugu Kawaguchi. Accordingly, we shall refer to the manifold associated with $\int F(x, x', \cdots, x^m)\,dt$ as a Kawaguchi space. Incidentally, a Euclidean

* For a discussion of Synge-Taylor geometry including the θ process reference may be made to J. H. Taylor, "A generalization of Levi-Civita's parallelism and the Frenet formulas," *Transactions of the American Mathematical Society*, vol. 27 (1925), p. 255 or J. L. Synge, "A generalization of the Riemannian line-element," *ibid.*, p. 61.

plane may be made the bearer of a Kawaguchi space in the following manner. Let there be given the set of all plane curves of class $C^{(m)}$, $x = x(t)$, $y = y(t)$ together with the set of normals to the x, y plane and let each of these curves be warped into the corresponding space curve; $x = x(t)$, $y = y(t)$, $z = \int_0^t (F^2(x, y; x', y'; \cdots; x^{(m)}, y^{(m)}) - (x'^2 + y'^2))^{1/2} dt$. Obviously, the length of arc of the part of one of these curves that joins the normals at $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ is given by the integral $\int_{P_1}^{P_2} F dt$ taken along the base curve.

In addition to the evident requirements as to differentiability etc., we shall assume in what follows that F satisfies two conditions, namely: (a) F is positive along each regular curve; (b) $F dt$ is invariant in functional form under an admissible parameter transformation.

6. *The vector T_r.* Obviously, if m is one, T_r is $F_{(1)r}$ and so our "tangent" vector is a generalization of the covariant tangent vector of Synge-Taylor geometry. Furthermore, in this case it is well known that (b) implies the identity $x'^r T_r = F$ and, as a matter of fact, this same implication has been established for $m = 2$.* Thus we are led to consider the situation in general.

As a preliminary we shall demonstrate that

$$\sum_{v=1}^{m+1} (-1)^{v-1} {}_mC_{v-1} [x^{(m+2-v)} F]^{(v-1)} = (-1)^m x' F^{(m)} \tag{3}$$

is an identity.

Proof. By the rule for differentiating products, we have

$$[x^{(m+2-v)} F]^{(v-1)} = \sum_{w=0}^{v-1} {}_{v-1}C_w x^{(m+1-w)} F^{(w)}.$$

Consequently, the coefficient of $x^{(m+1-w)} F^{(w)}$ in (3) is $\sum_{v=w+1}^{m+1} (-1)^{v-1} {}_mC_{v-1} \cdot {}_{v-1}C_w$, which, by virtue of the equality ${}_mC_{v-1}\, {}_{v-1}C_w = {}_mC_{m-w}\, {}_{m-w}C_{v-1-w}$ and the substitution $v = u + w + 1$ may be written ${}_mC_{m-w} \sum_{u=0}^{m-w} (-1)^{u+w} {}_{m-w}C_u$. But, by a well known property of the binomial coefficients this last expression is $(-1)^m \delta_w{}^m$ and the lemma is established. This accomplished, we turn to the

THEOREM. *A necessary condition for the invariance of functional form of $F(x, x', \cdots, x^{(m)}) dt$ under a parameter transformation is $x'^r T_r = F$.*

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* See H. V. Craig, "On parallel displacement in a non-Finsler space," *loc. cit.*, p. 133.

Proof. If F has the invariant property in question and $T_\cdot$ is any function of t then

$$(FT)' = \sum_{u=0}^{m} (x'^r T)^{(u)} F_{(u)r}.^*$$

From this by setting $T = t,\ t^2/2!$ etc. successively, we derive that

$$\sum_{u=v}^{m} {}_uC_v x^{(u-v+1)r} F_{(u)r} = \delta_v{}^1 F \qquad (v = 1, 2, \cdots, m).$$

Designating the left member of this equation with L_v we find by direct calculation, for small values of m) that $\sum_{v=1}^{m} (-1)^{v-1} v L_v{}^{(v-1)}$, which is obviously F, reduces identically in $F^{(w)}$ to $x'^r T_r$. If we assume that this reduction takes place for a given value of m and, in the case $m+1$, represent $T_r - (-1)^m (m+1) F_{\substack{\cdot\cdot m\\ m+1\, r}}$ with T_r', then we may write

$$\sum_{v=1}^{m+1} (-1)^{v-1} v\, L_v{}^{(v-1)} = \sum_{v=1}^{m} (-1)^{v-1} v \sum_{u=v}^{m} {}_uC_v [x^{(u-v+1)r} F_{(u)r}]^{(v-1)}$$
$$+ \sum_{v=1}^{m+1} (-1)^{v-1}\, v_{m+1}C_v (x^{(m+2-v)r} F_{(m+1)r})^{(v-1)}.$$

But the first term in the right member of the foregoing is by assumption $x'^r T_r'$, while the second can be put in the form

$$(m+1) \sum_{v=1}^{m+1} (-1)^{v-1}\ {}_mC_{m+1-v} (x^{(m+2-v)r} F_{(m+1)r})^{(v-1)}$$

which by (3) reduces identically to $(-1)^m (m+1) x'^r F_{\substack{\cdot\cdot (m)\\ m+1\, r}}$, and hence the theorem follows.

As a consequence of this we can so select the parameter that $x'^r T_r$ will maintain the value unity along any prescribed regular curve. Also, if we were to choose the quantities $F_{(m)r(m)s} + T_r T_s$ as the components of the metric tensor f_{rs} then, because $x'^r F_{(m)r(m)s} = 0$, we would have $x'^s f_{rs} = T_r$, and, with a properly selected parameter, $x'^r x'^s f_{rs} = 1$; $T_r T_s f^{rs} = 1$.

With regard to possible future developments based on T_r, we note that an obvious consequence of the definitions of T_r and S_r is the following: if $\{^j_r\}$ is any two index connection † then the extremal curves associated with

* See Adolph Kneser, *Lehrbuch der Variationsrechnung*, Braunschweig (1900), p. 195.

† I. e. an object which transforms as $x'^s\{^{\,j}_{sr}\}$, see L. P. Eisenhart, *Riemannian Geometry* (1926), p. 19. For a most general connection reference may be made to A. Kawaguchi, "Die Differentialgeometrie in der verallgemeinerten Mannigfaltigkeit,"

F are those for which the vectors θT_r ($\theta T_r = T'_r - T_j\{^j_r\}$) and S_r coincide. Should F and the connection be such that S_r is zero then the extremal curves may be characterized as auto-parallel and in this case we may conclude from (b) that $x'^r\theta T_r$ vanishes.† As a matter of fact such connections may be constructed. For if $\{^j_r\}$ is any two index connection and the generalized arc the parameter, then the quantities $\{^j_r\}^*$ defined by $\{^j_r\}^* = \{^j_r\} + x'^j S_r$ also constitute a connection. Evidently, this connection is such that the associated S^*_r vanishes, for S^*_r may be written $S_r - T_j x'^j S_r$.

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Rendiconti di Palermo, tomo 56 (1932), pp. 245-276; also see H. Hombu, "On a non-Finsler metric space," *Tohoku Mathematical Journal*, vol. 37 (1933), pp. 190-198.

† See H. V. Craig, "On the solution of the Euler equations for their highest derivatives," *Bulletin of the American Mathematical Society*, vol. 36 (1930), p. 560.